# KINEMATIC DERIVATION OF THE D'ALEMBERT WAVE EQUATION AND A NEW WAVE EQUATION WITH APPLICATIONS

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I derive harmonic wave equations from the kinematic requirement that the wave not change shape as it propagates. This requirement yields two first-order linear partial differential equations for waves propagating in opposite directions, which can be used to obtain two other wave equations: the usual second-order linear partial differential wave equation and a first-order quadratic wave equation. These two equations have a common set of solutions. The first-order equations can be expressed as one equation from which Poynting's theorem can be derived for electromagnetic waves propagating in vacuum. The first-order quadratic equation can be used to express the flow of energy for conservative systems. I apply this equation to waves in strings, electromagnetic waves in vacuum, and longitudinal waves. Since these equations are derived from kinematics, none depends on the nature of the oscillation or the medium through which it propagates, so long as it is conservative and kinetically homogeneous.

## I. Introduction

Mathematically, a harmonic wave can represented by a sinusoidal function of time and position or a linear superposition of such functions, since a single such sinusoidal function describes the propagation of a single simple harmonic oscillation (a "monochromatic" wave). The argument of each sinusoidal function is proportional to $x \pm vt$, where $x$ is the position along the wave, propagating parallel to the $x$ axis, where the oscillation is observed, $t$ the time variable for the oscillation, and $v$ the speed of the wave.

The harmonic wave equation can be derived dynamically or kinematically. Dynamically, the derivation depends on the type of medium in which the wave propagates. For example, the wave equation for elastic media depends on the displacement of the medium being proportional to the magnitude of the non-dissipative restoring force (or stress) that results from the displacement. Another example involves electromagnetic waves, which can be derived from electromagnetic theory. However, the kinematic derivation is not restricted to any particular type of medium. Kinematically, a harmonic wave does not change its shape as it propagates, and all harmonic waves are included in a kinematic derivation. A soliton also propagates without changing its shape, but there are restrictions on the shape of the waves, whereas a harmonic plane wave can be represented by a function of any shape that can be twice differentiated with respect to the argument $x \pm vt$, although some solutions have to be rejected on physical grounds.

The kinematic derivation results in two first-order differential equations that strictly apply to a purely kinematic propagation of a harmonic wave. These equations will not apply, in themselves, to situations where second-order differentiation is required (for example, a wave on a string). However, the kinematic derivation can be used to obtain not only D'Alembert's second partial derivative wave equation involving the forces acting on the medium through which the wave propagates, an equation familiar in textbooks, but also another equation, with the same solutions, involving energy propagation by the wave.

**II. Kinematic Derivation of the One-Dimensional Harmonic Wave Equation**

You can impose the requirement of constant shape for a plane wave moving in the positive *x* direction by requiring the displacement at a certain point on the wave in the medium through which the wave is propagating to be the same after the wave has moved a distance $\Delta x$ in a time $\Delta t = \Delta x/v$ where *v* is the wave speed. In equation form, the requirement that the displacement, call it *y*, at position $x+\Delta x$ and time $t+\Delta t$ is the same as that at position *x* and time *t* is

$$y(x+\Delta x, t+\Delta t) = y(x,t), \qquad (1)$$

where the wave speed is

$$v = \frac{\Delta x}{\Delta t}. \qquad (2)$$

See Fig. 1.

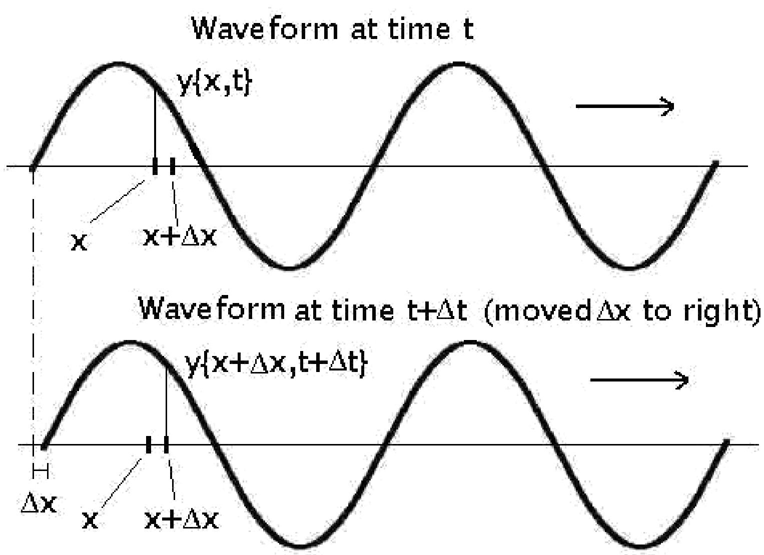


**Figure 1**

Since the displacement due to the passage of a wave, *y(x,t)*, is a function of two variables. The expansion of Eq. (1) in *x* and *t* to all orders is

$$y(x+\Delta x, t+\Delta t) = y(x,t) + \left(\frac{\partial y(x,t)}{\partial t}\right)\Delta t + \left(\frac{\partial y(x,t)}{\partial x}\right)\Delta x + \left(\frac{\partial^2 y(x,t)}{\partial x \partial t}\right)\Delta x \Delta t + \ldots \quad .(3)$$

If $\Delta x$ and $\Delta t$ are very small, the terms the the right of the first order terms will be small enough by comparison that they can be ignored in the approximation. So, if we stick to very small $\Delta' s,$ the expansion of *y(x,t)* becomes

$$y(x+\Delta x, t+\Delta t) \approx y(x,t) + \left(\frac{\partial y(x,t)}{\partial t}\right)\Delta t + \left(\frac{\partial y(x,t)}{\partial x}\right)\Delta x. \qquad (4)$$

Now insert the requirement that the displacement due to the wave at a position $x+\Delta x$ at a time $t+\Delta t = t+\Delta x/v$ is the same as the displacement at position *x* at the earlier time *t*. Replace $t+\Delta t$ by this equivalent in Eq. (5).

$$y(x+\Delta x, t+\Delta x/v) = y(x,t). \qquad (5)$$

Dividing through by $\Delta t$ in Eq. (4) and canceling out the equal terms by means of Eq. (5) gets you to the following equation.

$$\left(\frac{\partial y(x,t)}{\partial t}\right) + \left(\frac{\partial y(x,t)}{\partial x}\right)\frac{\Delta x}{\Delta t} \approx 0. \qquad (6)$$

Letting $\Delta x/\Delta t \rightarrow v$ as $\Delta t \rightarrow 0$ gives you

$$\frac{\partial y(x,t)}{\partial t} + v\left(\frac{\partial y(x,t)}{\partial x}\right) = 0. \qquad (7)$$

Any differentiable function of the form *f(x - vt)* is a solution to this equation, as can easily be verified.

One slight problem, however. Eq. (7) is the wave equation for harmonic waves moving in the positive x direction but not for those moving in the negative x direction. This is because I assumed a positively moving wave when I wrote Eq. (2). A wave moving in the negative direction would have a velocity -*v*, and you would have to rewrite Eq. (2) as

$$v = -\frac{\Delta x}{\Delta t}. \qquad (8)$$

to ensure a positive time interval. Going through all the steps again with the negative sign results in the wave equation for a negatively propagating wave,

$$\frac{\partial y(x,t)}{\partial t} - v\left(\frac{\partial y(x,t)}{\partial x}\right) = 0. \qquad (9)$$

To get a single equation good for both directions, you could let *v* be the linear velocity rather than the speed in Eq. (7) so *v* will take its sign. Eq. (7) would then have any harmonic wave with displacement *y* as a solution. Another possibility is to combine the two wave equations, (7) and (9) to get a single equation where *v* is the speed that includes waves propagating in either direction. One way to do this is to multiply the two together. This option will be discussed later.

Another way to combine the two equations is suggested by rewriting (7) as

$$\left[\frac{\partial}{\partial t} + v\left(\frac{\partial}{\partial x}\right)\right]y(x,t) = 0. \tag{10}$$

Written this way what is inside the square brackets can be considered an *operator*, operating on *y*. All you've done is factor out the derivative operations to construct the operator in square brackets, which is a linear combination of partial derivative operations. The same can be done for
Eq. (9),

$$\left[\frac{\partial}{\partial t} - v\left(\frac{\partial}{\partial x}\right)\right]y(x,t) = 0. \tag{11}$$

The operators in square brackets in Eqs. (10) and (11) commute because the partial derivatives do, and one will yield zero when operating on *y(x,t)* for whichever direction the wave is traveling. Therefore, why not operate on *y* with *both* operators, first one and then the other? (The order you do this does not matter, again because partial derivatives commute.) The result will be zero, because whether you operate on the function of a positively or negatively moving wave, one of the operations will give zero. Finally, you use the math identity $(a+b)(a-b) = a^2-b^2$ to combine the operators to give

$$\frac{\partial^2 y}{\partial t^2} - v^2\frac{\partial^2 y}{\partial x^2} = 0. \tag{12}$$

This is the harmonic wave equation for wave displacement *y*, good for waves traveling in both the positive and negative *x* direction. Functions of the form

$$f(x,t) = f(x \pm vt), \tag{13}$$

are solutions of this equation, provided they can be differentiated at least twice. A "monochromatic" harmonic wave of the form

$$y(x,t) = A cos[k(x \pm vt)+\varphi], \tag{14}$$

where *A* is the wave amplitude, *k* is a constant (called the "wave number" and equal to 2π divided by the wavelength), and $\phi$ is an arbitrary (phase) angle, are solutions of the equation with a single frequency (= $kv/2\pi$ ) and wavelength (= $2\pi/k$ ).

Now examine the first option mentioned above, that is, multiply Eqs. (7) and (9) together. The equation you get is

$$\left(\frac{\partial y}{\partial t}\right)^2 - v^2\left(\frac{\partial y}{\partial x}\right)^2 = 0. \tag{15}$$

It is not hard to show that this equation has the same solutions as Eq. (12), that is, any function *singly* differentiable of the form $f(x \pm vt)$ is a solution of Eq. (15). Therefore, it is just as viable a wave equation as Eq. (12). Furthermore, functions that are only differentiable once, not twice, qualify mathematically as solutions, in contrast to the double-differentiation requirement for Eq. (15).

However, to be a physically acceptable solution in situations like the propagation of waves in strings, the function must be doubly differentiable, since the wave function must satisfy Eq. (12) as well, in order for the acceleration of *y* to be defined.

## III. The Three-Dimensional Plane-Wave Equation

The three dimensional equations analogous to Eqs. (10) and (12) for a medium where the velocity is constant are

$$\frac{\partial \psi}{\partial t} \pm \boldsymbol{v}\cdot\nabla\psi = 0, \tag{16}$$

where you have to use something other than "*y*" for the wave displacement if *y* is used as one of the three position variables. (I use *x*, *y*, and *z* here.) I have chosen to use $\psi$ to stand for the wave displacement. Here ***v*** is the velocity vector in the "positive" direction (making ***-v*** the velocity vector in the "negative" direction), the symbol $\nabla$ is the gradient operator,

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z}, \tag{17}$$

and "$\cdot$" denotes the "scalar product" or "dot product". $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors in the *x*, *y*, and *z* directions, respectively.

You can combine Eqs. (16) in the same way as before to get two wave equations. First you define two operators

$$\left(\frac{\partial}{\partial t} \pm \boldsymbol{v}\cdot\nabla\right), \tag{18}$$

from Eq. (16). (It is interesting to note that, if you let v take its sign allowing you to include only the positive sign in Eq. (20), the expression becomes that of the advective time derivative. This means $\psi$ acts like a scalar advective field in the absence of forces. That is, a wave consists of a “flow” of amplitude, as well as the oscillation at a particular point on the wave.) Take the product of the two operators in Eq. (18) and operate on the displacement:

$$\left[\frac{\partial^2}{\partial t^2} - (\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\nabla)\right]\psi = \left[\frac{\partial^2}{\partial t^2} - (\boldsymbol{v}\cdot\nabla)^2\right]\psi = 0. \tag{19}$$

This equation includes the case where the velocity is not parallel to the gradient of the displacement. You can now use a vector identity as follows to get the usual three-dimensional wave equation where the velocity is parallel to the gradient. First handle the term,

$$(\boldsymbol{v}\cdot\nabla)\psi = \nabla\cdot(\boldsymbol{v}\,\psi) - \psi\nabla\cdot\boldsymbol{v} = \nabla\cdot(\boldsymbol{v}\,\psi), \tag{20}$$

in Eq. (19), noting that the divergence of ***v*** vanishes since ***v*** is a constant. The spatial operation in Eq.

(19) now becomes, applying the vector identity $\nabla(\nabla\cdot\boldsymbol{a}) = \nabla\times(\nabla\times\boldsymbol{a}) + \nabla^2\boldsymbol{a}$,

$$\boldsymbol{v}\cdot\nabla(\nabla\cdot(\boldsymbol{v}\psi)) = \boldsymbol{v}\cdot\nabla\times(\nabla\times\boldsymbol{v}\psi) + v^2\nabla^2\psi. \tag{21}$$

In this equation, the first term on the right hand side is zero from the following analysis. Write the parenthetical part of this term using a vector identity as

$$\nabla\times\boldsymbol{v}\psi = (\nabla\psi)\times\boldsymbol{v} + \psi\nabla\times\boldsymbol{v} = 0. \tag{22}$$

The second term on the right is zero due to $\boldsymbol{v}$ being a constant. The first term vanishes if you assume that $\nabla\psi$ and $\boldsymbol{v}$ are parallel, that is, that the wave propagates in the direction of the gradient of the displacement. Putting this all together gives

$$(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\nabla)\psi = v^2\nabla^2\psi = v^2\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right]. \tag{23}$$

Substituting this into Eq. (21) yields the three-dimensional wave equation,

$$\frac{\partial^2\psi}{\partial t^2} - v^2\nabla^2\psi = 0. \tag{24}$$

Then, if you choose your axes such that the propagation direction is along one of them, say the x axis, Eq. (24) reduces to

$$\frac{\partial^2\psi}{\partial t^2} - v^2\frac{\partial^2\psi}{\partial x^2} = 0. \tag{25}$$

To get the second wave equation, operate on the displacement with each operator in Eq. (16) independently, the multiply the results together. This gives

$$\left(\frac{\partial\psi}{\partial t}\right)^2 - (\boldsymbol{v}\cdot\nabla\psi)^2 = 0. \tag{26}$$

If you make the assumption that the wave propagates in the same direction as the gradient of $\psi$ (that is, perpendicular to the wavefront), Eq. (26) can be written as

$$\left(\frac{\partial\psi}{\partial t}\right)^2 - v^2(\nabla\psi)^2 = 0. \tag{27}$$

Like its 1-D counterpart, this equation can be applied to the energy carried by the wave. The solutions to Eqs. (19) and (26) are of the form

$$\psi(\boldsymbol{r},t) = \psi[\boldsymbol{k}\cdot(\boldsymbol{r}+\boldsymbol{v}t)], \tag{28}$$

where $\boldsymbol{k}$ points in the direction of propagation of the wave, $\boldsymbol{r}$ is the position vector of the wave displacement, t is the time, and $\boldsymbol{v}$ is the velocity of the wave. The only requirement for this function to be a solution of these equations is that the appropriate derivatives exist. Once again, some solutions have to be rejected on physical grounds. These waves are called *plane waves*, because in three dimensions the wavefronts are flat planes. (A two-dimensional example of a plane wave is a series of parallel ocean swells.) If $\boldsymbol{k}$ is parallel or antiparallel to $\boldsymbol{v}$, then the time-dependent part of the argument in Eq. (28) is $\pm kvt$ where $k$ and $v$ are the magnitudes of the two vectors, $\boldsymbol{k}$ and $\boldsymbol{v}$, respectively. This adjustment would make Eq. (28) a solution of Eqs. (24) and (27).

Since Eqs. (24) and (27) can be applied to the Cartesian components of a vector displacement, they can be written as before, with the scalar displacement replaced by a vector displacement. The corresponding vector equations therefore are, assuming each component propagates with the same speed,

$$\frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} - v^2 \nabla^2 \boldsymbol{\psi} = 0, \tag{29}$$

and

$$\left(\frac{\partial \boldsymbol{\psi}}{\partial t}\right)^2 - v^2\left[\left(\frac{\partial \boldsymbol{\psi}}{\partial x}\right)^2 + \left(\frac{\partial \boldsymbol{\psi}}{\partial y}\right)^2 + \left(\frac{\partial \boldsymbol{\psi}}{\partial z}\right)^2\right] = 0. \tag{30}$$

If you take $\boldsymbol{v}$ to be in the direction of one of the Cartesian axes, say the $x$ axis, this Eq. (30) becomes

$$\left(\frac{\partial \boldsymbol{\psi}}{\partial t}\right)^2 - v^2\left(\frac{\partial \boldsymbol{\psi}}{\partial x}\right)^2 = 0. \tag{31}$$

**IV. Force and Energy of Waves in Strings from Both Wave Equations**

The first term in Eq. (14) is readily recognized as the oscillatory acceleration of an element of the medium at a certain position $x$ through which the wave is traveling. For mechanical waves the displacement element is that of mass. For example, if the medium is an ideal elastic string, the term describes the transverse acceleration of a point mass at position $x$ along the string. Newton's second law of motion says the acceleration of a mass times that mass equals the (net) force acting on the mass. Therefore, if you multiply the mass per unit length times the first term in Eq. (12), you get the net transverse force per unit length acting on this linear medium. Applying this idea by multiplying Eq. (12) through by the mass per unit length, call it $\mu$, and moving the second term to the right side, you get

$$\mu \frac{\partial^2 y}{\partial t^2} = \mu v^2 \frac{\partial^2 y}{\partial x^2}. \tag{32}$$

Since the term on the left is the transverse force per unit length, the term on the right has to equal the transverse force per unit length. (For waves traveling through a continuous medium, you would have force per unit area in the form of pressure and stress instead of force per unit length.) Call the net transverse force per unit length "*f*". From Eq. (32) the equation for *f* is

$$f(x,t) \ = \ \mu v^2 \frac{\partial^2 y}{\partial x^2}. \tag{33}$$

For an elastic string, it is known that the wave speed equals the square root of the tension in the string, *F*, divided by the mass per unit length of the string $\mu$ or,

$$v \ = \ \sqrt{\frac{F}{\mu}}. \tag{34}$$

Therefore, the net transverse force per unit length along a string is given by

$$f(x,t) \ = \ F \frac{\partial^2 y}{\partial x^2}. \tag{35}$$

By geometry the transverse force (not the *net* transverse force) in a string acting opposite to the direction of propagation (see Fig. 2) is

$$F_T \ = \ -F \frac{\partial y}{\partial x}. \tag{36}$$

It is a curious property of sinusoidal waves that the maximum net transverse force *per unit length* occurs where the transverse force $F_T$ is zero and zero where the transverse force is maximum. This is because the net transverse force is the difference in the transverse force across a segment of string, and the difference is zero at the nodes and its magnitude approaches maximum at the top of the peaks and bottom of the troughs.

Eq. (7) can be used to get the power transmitted in the direction of wave propagation in a string by the transverse force. To see how power is transmitted, imagine cutting, in the mind's eye, a sinusoidal wave propagating in the positive x direction (to the right) on a string at the front end of an approaching peak and replacing the left part of the string by the forces it exerts on the right part. (See Fig. 2. The dot above the ***y*** in this figure stands for the partial time derivative of that variable.)

Now power is the scalar (dot) product of force and velocity, that is $p \ = \ \boldsymbol{F} \cdot \boldsymbol{v}$. The velocity in this case is the transverse velocity $\partial y / \partial t \ ( \ = \ \dot{y})$ (not the wave velocity), which falls in either the positive or negative *y* direction, therefore only the *y* component of the tension, that is, the transverse force of Eq. (36), contributes to the scalar product. On the leading edge of a peak, the transverse velocity is in the positive direction, which means power is imparted to the right part of the string. On the trailing edge of a peak both the transverse force and velocity are negative, and once again power is positive and transmitted to the right part of the string. From these considerations the power transmission is given by

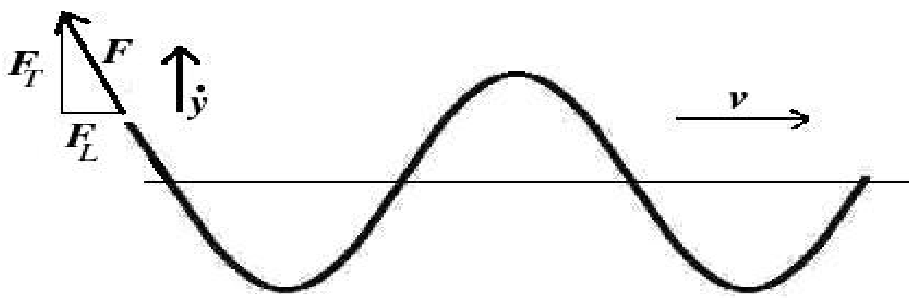


**Figure 2**

$$p \;=\; F_T \frac{\partial y}{\partial t} \;=\; -F\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \;=\; Fv\left(\frac{\partial y}{\partial x}\right)^2, \tag{37}$$

where Eq. (7) has been used letting *v* take its sign: plus for propagation in the positive direction and minus otherwise, so that the power is shown to be transmitted in the propagation direction.

If the wave is a harmonic wave of the form of Eq. (16), the net force per unit length varies as

$$f(x,t) \;=\; -k^2 FAcos[k(x \pm vt) + \varphi] \;=\; -k^2 Fy(x,t). \tag{38}$$

Making an analogy with a mass-spring system, where the restoring force is the negative of the spring constant (aka spring stiffness factor) times the displacement, you see that the "spring constant" equals the wave number squared times the ambient tension, that is, $k^2 F$. This means the shorter the wavelength and the greater the tension, the stiffer the "spring" acting to restore the element of string to its equilibrium position.

The net transverse force can also be obtained from Eq. (15), the first-order quadratic wave equation. The first term in this equation is the square of the speed of an element of the medium in oscillatory motion. If you multiply this by one-half times mass per unit length, the term becomes the (non-relativistic) kinetic energy per unit length of the medium. If the medium does not dissipate the wave's energy, it is conservative, and if the wave speed does not depend on wavelength, the wave will not change shape. In addition, if the wave speed is the same in all directions, the medium is isotropic. For a conservative medium, the kinetic plus potential energy per unit length must be constant. Also, the potential energy must be a function of position only, which means it is a function of the displacement *y*. Multiplying Eq. (15) through by $\mu/2$ you get

$$\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 = 0. \tag{39}$$

The first term in the equation is the kinetic energy per unit length. What is the second term? Say the total energy per unit length $= e$. Add this to both sides to get

$$\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 + e = e. \tag{40}$$

This shows that the term

$$-\frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2 + e, \tag{41}$$

equals the potential energy per unit length. However, since the system is conservative, the zero of the potential energy can be reassigned arbitrarily. Therefore, define the zero of potential energy to be such that the total energy is zero. (This means the potential energy is always negative and is equal to the negative of the always positive kinetic energy.) If you redefine the potential energy this way, then the potential energy per unit length (call it "*u*"), is

$$u(x,t) = -\frac{1}{2}\mu v^2\left(\frac{\partial y}{\partial x}\right)^2, \tag{42}$$

where

$$w(x,t) = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2, \tag{43}$$

is the kinetic energy per unit length.

The relationship between transverse force per unit length and potential energy per unit length is given by the usual formula

$$f(x,t) = -\frac{\partial u}{\partial y}. \tag{44}$$

The derivative is taken with respect to $y$, since the transverse force is a function of $y$. It is not hard using the chain rule to show that the Eq. (44) gives Eq. (33) when $u$ is given by Eq. (42).

From another point of view net power expended at a certain point along the string should be the partial time derivative of Eq. (41). Doing this you get

$$\frac{\partial w}{\partial t} = \left(\mu\frac{\partial^2 y}{\partial t^2}\right)\left(\frac{\partial y}{\partial t}\right) = \left(\mu v^2\frac{\partial^2 y}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right). \tag{45}$$

The far right term has to be net transverse force times the net transverse speed, showing that the net transverse force is the same as is given in Eq. (33).

The potential energy in a length of the string *L*, is given, using Eq. (42), by

$$PE \;=\; -\frac{1}{2}\mu v^2 \int_L \left(\frac{\partial y}{\partial x}\right)^2 dx\,. \tag{46}$$

The force (tension) in the longitudinal (*x*) direction (not to be confused with the ambient tension present before the wave motion) due to this potential energy is then given by

$$F_L \;=\; -\frac{PE}{\partial x} \;=\; \frac{1}{2}\mu v^2 \left(\frac{\partial y}{\partial x}\right)^2. \tag{47}$$

This longitudinal force is in addition to the ambient tension. It is zero when the slope of the spring is zero and maximum when the slope is maximum. The presence of this longitudinal, time-varying tension means there is a very slight longitudinal oscillation in the string.

Using Eq. (34) you can express this force in terms of the ambient tension as

$$F_L \;=\; \frac{1}{2}F\left(\frac{\partial y}{\partial x}\right)^2. \tag{48}$$

Comparing Eq. (48) to the transverse force in Eq. (36) and realizing that the slope of a vibrating string is generally small, you can see that this oscillation is negligible compared to the transverse oscillation.

To get the total tension in the string, you add the additional longitudinal force to the ambient tension to get the total longitudinal component; then you combine the transverse force to this longitudinal component by vector addition. The result is

$$F_{total} \;=\; \sqrt{(F+F_L)^2+F_T^2} \;=\; F\sqrt{1+2\left(\frac{\partial y}{\partial x}\right)^2}\,, \tag{49}$$

ignoring the presumably negligible fourth power of the slope.

## V. Electromagnetic Waves and Radiation Pressure

The electromagnetic plane-wave equation from Maxwell's electromagnetic theory for waves moving through a vacuum is exactly the same as Eq. (29) derived in Section III from kinematic considerations with one exception. There are two vector components to an electromagnetic wave, not just the one component of displacement as, for example, in the case of a wave on a string. One is the electric field ***E*** and the other the magnetic field ***B***. The equation for each is the same as Eq. (29) with the two displacements, ***E*** and ***B***. The two equations are

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - c^2\left[\frac{\partial^2 \boldsymbol{E}}{\partial x^2} + \frac{\partial^2 \boldsymbol{E}}{\partial y^2} + \frac{\partial^2 \boldsymbol{E}}{\partial z^2}\right] = 0, \tag{50}$$

and

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} - c^2\left[\frac{\partial^2 \boldsymbol{B}}{\partial x^2} + \frac{\partial^2 \boldsymbol{B}}{\partial y^2} + \frac{\partial^2 \boldsymbol{B}}{\partial z^2}\right] = 0, \tag{51}$$

where the speed $v$ is replaced by the speed of light, traditionally symbolized by "$c$".

You lose no generality for the following work if you take the electric-field wave to be moving in the positive or negative $x$ direction with the field $\boldsymbol{E}$ vibrating in some direction, arbitrary except that, for a plane wave in vacuum, it must vibrate in a direction perpendicular to the propagation direction, $x$. Hence, you can assume it is vibrating in the $y$ direction, or the $z$ direction, or some direction in between. If you take the electric field to lie along an axis to make it a one-dimensional vector and call it "$E$" without the bold font, then Eq. (50) becomes

$$\frac{\partial^2 E}{\partial t^2} - c^2\frac{\partial^2 E}{\partial x^2} = 0, \tag{52}$$

with an identical equation for the magnetic field B. (By electromagnetic theory these fields have to be at right angles for a plane wave traveling in a vacuum.)

What I intend to do is to apply what was covered earlier to the wave equation for the electric field, Eq. (52). A similar result would apply to the wave equation of the magnetic field. A three-dimensional electric field monochromatic plane wave moving in the positive $x$ direction is given by the equation

$$E = E_o \cos(kx - \omega t + \varphi). \tag{53}$$

The constant $\phi$ is the phase angle, which is arbitrary in this case and can be assigned a value of zero. $E_o$ is the amplitude of the wave, $k$ is the wave number and equal to $2\pi$ divided by the wavelength $\lambda$ ($k = 2\pi/\lambda$), and $\omega$ is the angular frequency, equal to $2\pi$ times the ordinary frequency, $f$ ($\omega = 2\pi f$). This wave is a solution of Eq. (52), since it can be written as a function of "$x - ct$" as follows.

$$E = E_o \cos[k(x - ct)], \tag{54}$$

where $c = \omega/k = \lambda$.

The electric field wave equation can be written in the alternate form of Eq. (15).

$$\left(\frac{\partial E}{\partial t}\right)^2 - c^2\left(\frac{\partial E}{\partial x}\right)^2 = 0. \tag{55}$$

From electromagnetic theory, the time-varying energy density of the wave, call it $u(x,t)$, is given by the formula

$$u(x,t) \ = \ \epsilon_o E(x,t)^2 . \tag{56}$$

This equation includes the energy in both the electric and magnetic fields.

For the solution given in Eq. (54),

$$u(x,t) \ = \ \epsilon_o E_o^2 \cos^2(kx - \omega t) . \tag{57}$$

Note that the time-varying energy density goes from zero to a maximum of $\epsilon_o E_o^2$. What happens to the energy contained in the fields when *u* goes to zero? By electromagnetic theory, this is the energy propagated forward by the plane wave according to conservation of energy.

Electromagnetic theory doesn't really say *what* the energy density is (you could add an arbitrary constant to Eq. (56)), just that the loss or gain of energy density in a given volume has to be balanced by the flow of energy out of or into that volume such that the conservation of energy is observed. (This is known as Poynting's theorem.) But, for convenience, for the time being I will accept Eq. (56) as the total energy density of an electromagnetic wave.

You can turn the electric field wave equation (55) into an energy density wave equation by multiplying through by $\epsilon_o/\omega^2$. This changes the units of each term to joules per cubic meter. You get

$$\left(\frac{\epsilon_o}{\omega^2}\right)\left(\frac{\partial E}{\partial t}\right)^2 - \left(\frac{\epsilon_o c^2}{\omega^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 \ = \ 0 . \tag{58}$$

Or, since $\omega/c \ = \ k$,

$$\left(\frac{\epsilon_o}{\omega^2}\right)\left(\frac{\partial E}{\partial t}\right)^2 - \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 \ = \ 0 \tag{59}$$

Following the same arguments leading to Eq. (40), you can add the "total" energy density to both sides of the equal sign and write Eq. (59) as

$$\left(\frac{\epsilon_o}{\omega^2}\right)\left(\frac{\partial E}{\partial t}\right)^2 - \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 + \epsilon_o E_o^2 \ = \ \epsilon_o E_o^2 . \tag{60}$$

You might be tempted to treat the first term of Eq. (60) as analogous to the kinetic energy density as in Eq. (40). Then the second and third terms on the left of the equal sign constitute the analogy of the potential energy density. The sum of the terms on the left equals the total energy density on the right of the equal sign. However, once again you have the arbitrariness of the zero level of the potential energy. After all, you can add *any* energy density to both the left and right side of the equal sign, and the equation is still valid. If you take the total energy to be zero as explained in the text surrounding Eq. (40), you get back to Eq. (59).

If the second term in Eq. (59) is truly the potential energy density, then integrating it over a volume should produce the potential energy contained in that volume, that is,

$$PE \ = \ - \int_V \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 dV. \tag{61}$$

Then the "force" that results from the release of this potential energy should be the negative of the gradient of Eq. (61), the theoretical relationship between force and potential energy. Now *E* is a function only of *x* and *t*, so

$$F \ = \ -\frac{\partial PE}{\partial x} \ = \ \int_A \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 dydz \ = \ \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 A, \tag{62}$$

where *A* is an arbitrary area perpendicular to the direction of propagation, and the wave is assumed to be propagating in the direction of positive *x*. The force per unit area, which can be identified with the radiation pressure, is then given by

$$\frac{F}{A} \ = \ \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2. \tag{63}$$

Apply this to the monochromatic wave solution of Eq. (54), averaging in space and time over one period and one wavelength by the well-known formula,

$$\frac{1}{\pi}\int_0^\pi \sin^2\theta \, d\theta \ = \ \frac{1}{2}, \tag{64}$$

which gives you the radiation pressure due to the electric and magnetic fields as

$$p_E \ = \ \frac{1}{2}\epsilon_o E_o^2. \tag{65}$$

This is the radiation pressure if the energy of the electromagnetic radiation is absorbed by a surface on which it impinges. If the energy is reflected, the pressure would have to be twice this, or

$$p_E \ = \ \epsilon_o E_o^2. \tag{66}$$

This analysis appears to support the energy interpretation of Eq. (59).

## VII. From the Wave Equation to Poynting's Theorem

Retrieve Eq. (16), this time letting ***v*** take its sign.

$$\frac{\partial \psi}{\partial t} + \boldsymbol{v}\cdot\nabla\psi = 0, \tag{67}$$

Multiply through by $\psi$. The equation can then be expressed as follows,

$$\frac{\partial \psi^2}{\partial t} + \nabla\cdot(\boldsymbol{v}\psi^2) = 0. \tag{68}$$

where the vector identity

$$(\boldsymbol{v}\cdot\nabla)\psi^2 = \nabla\cdot(\boldsymbol{v}\psi^2) - \psi^2\nabla\cdot\boldsymbol{v} \rightarrow \nabla\cdot(\boldsymbol{v}\psi^2). \tag{69}$$

and the fact $\boldsymbol{v}$ is constant have been used.

For a plane electromagnetic wave with an electric field "displacement" *E*, choosing the vector to lie along an axis, Eq. (68) becomes

$$\frac{\partial E^2}{\partial t} + \nabla\cdot(\boldsymbol{c}E^2) = 0. \tag{70}$$

where $\boldsymbol{c}$ is the *velocity* of light, that is, it has the magnitude of the speed of light, *c*, and the direction in which the light is going.

This is actually a form of Poynting's theorem for an electromagnetic plane wave in vacuum. To see this recall from electromagnetic theory that such a wave travels in the direction of $\boldsymbol{E}\times\boldsymbol{B}$. Since this is the direction of $\boldsymbol{c}$, $\boldsymbol{c}$ can be written as

$$\boldsymbol{c} = \left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{|\boldsymbol{E}\times\boldsymbol{B}|}\right)c. \tag{71}$$

Replace $\boldsymbol{c}$ in Eq. (70) by this expression and note that for an EM wave in vacuum, $|\boldsymbol{E}\times\boldsymbol{B}| = EB = E^2/c$. Also multiply the equation through by the permittivity of free space, $\epsilon_o$, noting that the energy density of the wave is given by Eq. (58). With these changes you get

$$\frac{\partial u}{\partial t} + \epsilon_o c^2\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B}) = 0. \tag{72}$$

Since Poynting's vector is

$$\boldsymbol{S} = \epsilon_o c^2(\boldsymbol{E}\times\boldsymbol{B}), \tag{73}$$

you get Poynting's theorem for an EM plane wave in vacuum in its usual form from Eq. (72) as

$$\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{S} = 0. \tag{74}$$

The first term in Eq. (74) is the time rate of change of the electromagnetic energy density at a given point in space. This implies that the second term is the flow of energy away from that point in order for energy to be conserved.

If you then use the fact that electromagnetic energy density per unit time can be turned into mechanical energy density per unit time by the action of the electric field on moving charges through the term $\boldsymbol{J}\cdot\boldsymbol{E}$, where ***J*** is current density, then you can generalize Eq. (74) to

$$\frac{\partial u}{\partial t} + \nabla\cdot\boldsymbol{S} = -\boldsymbol{J}\cdot\boldsymbol{E}. \tag{75}$$

This is the general form of Poynting's equation, although here it is not the result of a general derivation. However, it does seem reasonable that the electromagnetic energy change in the vacuum should be due to the interaction of charges with the electric field.

Instead of proceeding to Poynting's theorem, you can retrieve Eq. (68), replace $\psi$ with E, multiply through by with $\epsilon_o$, and get the advective derivative of u to be zero, as follows.

$$\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u = 0. \tag{76}$$

For an electromagnetic wave in vacuum, therefore, this is equivalent to Poynting's theorem.

As an aside, you can go back to Eq. (59),

$$\left(\frac{\epsilon_o}{\omega^2}\right)\left(\frac{\partial E}{\partial t}\right)^2 - \left(\frac{\epsilon_o}{k^2}\right)\left(\frac{\partial E}{\partial x}\right)^2 = 0 \tag{59}$$

replace *E* with $(u/\epsilon_o)^{1/2}$, perform some differentiation and come up with

$$\left(\frac{\partial u}{\partial t}\right)^2 - c^2\left(\frac{\partial u}{\partial x}\right)^2 = 0, \tag{77}$$

showing that the energy density of an electromagnetic wave also obeys the alternative wave equation.

## VII. Longitudinal Waves

The vibration of the medium through which these waves travel is parallel to the propagation velocity of the wave. I consider plane waves traveling through a three-dimensional medium that retain their shape and therefore are solutions to the harmonic wave equation. It is sufficient to specialize, with no loss of generality, to waves traveling in the *x* direction, which means the vibration will be parallel to the *x* direction. This vibrational displacement has to be called something other than x, and here I choose to call it *s*. Then the usual wave equation is given by

$$\frac{\partial^2 s}{\partial t^2} - v^2 \frac{\partial^2 s}{\partial x^2} = 0, \tag{78}$$

with the alternative wave equation given by

$$\left(\frac{\partial s}{\partial t}\right)^2 - v^2 \left(\frac{\partial s}{\partial x}\right)^2 = 0. \tag{79}$$

An element of mass subject to acceleration as the wave goes by is given by $\rho A \delta x,$ where $\rho$ is the mass density of the medium, A is an area parallel to the wave front and perpendicular to the velocity, and $\delta x$ is a (time-independent) “infinitesimal” length perpendicular to A.

Multiplying both sides of Eq. (78) by the element of mass, you get

$$\rho A \delta x \frac{\partial^2 s}{\partial t^2} = \rho A \delta x v^2 \frac{\partial^2 s}{\partial x^2}. \tag{80}$$

Of course $\rho$ is a function of *x* and *t* as the wave moves by. If $\rho_o$ is the unperturbed mass density, then

$$\rho \approx \rho_o + \frac{\partial \rho}{\partial x} \delta x + \frac{\partial \rho}{\partial t} \delta t, \tag{81}$$

to first order. The changes in mass density enter as second-order corrections in Eq. (80) and can be ignored for small changes in $\rho,$ which can be replaced by $\rho_o.$ The left-hand side of Eq. (80) is the force $\delta F$ acting on the element of mass; therefore, so is the right-hand side. The force per unit area acting on the element of mass, in other words, the pressure, can be found by dividing the equation by the area *A*. However, it must be recognized that a force in the positive *x* direction means the pressure must be lower in the positive *x* direction. Hence you have

$$\delta P = -\frac{\delta F}{A} = -\rho_o v^2 \frac{\partial^2 s}{\partial x^2} \delta x, \tag{82}$$

as the small quantity of pressure acting on the element of mass. Since this equation was obtained holding *t* constant, it can be written as

$$\frac{\partial P}{\partial x} = -\rho_o v^2 \frac{\partial^2 s}{\partial x^2}. \tag{83}$$

Then the total pressure is gotten by integration.

$$P = P_o + P' = P_o - \rho_o v^2 \frac{\partial s}{\partial x}, \tag{84}$$

where $P_o$ is the ambient pressure before the wave moves through, and *P'* is the difference between that and the total (instantaneous) pressure.

The pressure in a harmonic longitudinal wave can also be obtained from Eq. (79). Multiply both sides of this equation by one half times the element of mass used earlier. The first term will then be the kinetic energy of the mass element and the second will be the associated potential energy. The equation is

$$\frac{1}{2}\rho_o A\,\delta x\left(\frac{\partial s}{\partial t}\right)^2 \;-\; \frac{1}{2}\rho_o A\,v^2\,\delta x\left(\frac{\partial s}{\partial x}\right)^2 \;=\; 0. \tag{85}$$

The potential energy per unit volume, $u$, is therefore, adjusting the zero level appropriately,

$$u \;=\; -\frac{1}{2}\rho_o v^2\left(\frac{\partial s}{\partial x}\right)^2. \tag{86}$$

The force per unit volume in the x direction should therefore be

$$f_V \;=\; -\frac{\partial u}{\partial s} \;=\; -\frac{\partial u}{\partial x}\frac{\partial x}{\partial s} \;=\; -\frac{\partial}{\partial x}\left[-\frac{1}{2}\rho_o v^2\left(\frac{\partial s}{\partial x}\right)^2\right]\frac{\partial x}{\partial s} \;=\; \rho_o v^2\frac{\partial^2 s}{\partial x^2} \;=\; -\frac{\partial P}{\partial x}. \tag{87}$$

The pressure gradient is seen to be the same as given in Eq. (83).

The intensity, $I$, of the longitudinal wave can be found from Eq. (87). From that equation the work done on a small volume of the medium is given by the force per unit volume times the element of volume on which it acts times the displacement produced.

$$\delta W \;=\; f_V A\,\delta x\,\delta s \;=\; -\frac{\partial P}{\partial x}\delta x\,A\,\delta s. \tag{88}$$

Since the intensity is the work done per unit time per unit area, you get

$$I \;=\; \frac{1}{A}\frac{\delta W}{\delta t} \;=\; -\frac{\partial P}{\partial x}\delta x\frac{\delta s}{\delta t} \;=\; P'\frac{\partial s}{\partial t}. \tag{89}$$

Hence the intensity is equal to the difference, $P'$, between the total pressure, $P$, and the ambient pressure, $P_o$, times the vibrational speed of the medium at that point.

## VIII. Conclusion

For a wave that propagates without a change in its shape, it is possible to derive, without reference to the cause of the oscillation, the usual wave equation, Eq. (29) ), that depends on the second derivatives of the oscillating quantity and Eq. (31) that depends on the squares of the first derivatives. Both types of equations have common solutions and are derived from first-order linear equations based only on a kinematic analysis, Eq. (16). This equation is shown to lead to Poynting's theorem in vacuum electromagnetic theory. The second order differential equation, as is well known, expresses the propagating oscillation of some amplitude, for example, Newton's second law for oscillations in conservative material media. The first-order quadratic equation expresses energy flow in conservative systems. Using this equation I have derived the potential energy per unit length for a wave on a string, the net transverse force for a wave in a string, radiation pressure due to an electromagnetic wave in vacuum, and the pressure and intensity of a plane longitudinal wave in a three-dimensional conservative medium. Although I assumed waves where the propagation speed was independent of frequency, you can apply this analysis to waves where the speed is dependent on frequency so long as you limit the analysis to spectral ("monochromatic") components of a composite wave.