Hidden momentum: A misapplication of the center of energy theorem

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Abstract

The inference of "hidden" linear momentum in the static electromagnetic fields of models studied in the literature is based on the theorem in relativity that a closed system whose center of energy is at rest has zero total momentum. The problem is that the systems examined and thought to have hidden momentum are not at rest in their original rest frames, so applying this theorem to them as if they were stationary in the original rest frame is erroneous. This error is due to ignoring the Lorentz forces that arise when putting these systems together and the energy expended in the assembly. In this paper I show, for a number of models examined in the literature, that hidden momentum does not exist if you take into account how the system is assembled. In this paper I recalculate the momenta of several systems appearing in the literature and show they do not contain hidden momentum.

I. INTRODUCTION

The linear momentum residing in an electromagnetic field is given, in the Lorentz formulation, by the following volume integral of the electromagnetic linear momentum density.

$$P_{em} = \epsilon_o \int_V (\boldsymbol{E} \times \boldsymbol{B}) dV, \tag{1}$$

where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ the magnetic field. In a system with static electromagnetic fields, this momentum, as is generally thought[1,3], must be balanced by an equal amount of relativistic hidden momentum somehow mechanically present in the system. "Hidden momentum", if it actually exists, is thus a rather obscure property of these systems. It appears the basic assumption that leads to the idea of hidden momentum is that the center of energy of the systems examined is stationary. But these systems are not stationary with respect to their original rest frame. Overlooked in models in which hidden momentum is inferred is the lack of appreciation of the effect of forces required to assemble the models originally. When the mechanical momentum imparted to the model in its assembly is taken into account, the total momentum, electromagnetic plus mechanical, is conserved without the need to postulate a hidden form.

In this paper I have considered a number of models that have appeared in the literature and show they contain no hidden momentum when the assembly of the models is included in the calculation of total momentum. The examples I examine are:

- An ideal electric dipole in a uniform magnetic field[3].
- A sphere carrying a surface charge producing a dipole electric field outside the sphere coupled with a uniform axial surface current on the sphere[1,5].
- A magnetic moment in the vicinity of a point charge[4].

II. AN IDEAL ELECTRIC DIPOLE IN A UNIFORM EXTERNAL MAGNETIC FIELD

Babson *et al*[3] calculate the electromagnetic momentum and impulses for an electric dipole in a uniform magnetic field using two models. In one the magnetic field is produced by a uniformly charged spherical shell centered at the origin of a three-dimensional Cartesian coordinate system. The other has the electric dipole inside a long solenoid. The purpose of these models is to represent a capacitor in a magnetic field. Neither of these models contains hidden momentum as will be shown below. In fact the argument against hidden momentum is the same for both models.

Looking at the first model, at the origin and center of the shell is a point electric dipole with its dipole moment pointing in the positive y direction. The shell is rotating about the z axis such that a uniform current is set up which produces a magnetic field. Inside the shell the magnetic field is uniform and points in the positive z direction. Outside the shell the field is that of a magnetic dipole with its moment lying on the z axis and pointing in the positive z direction. They calculate the field momentum due to this rotating sphere and the point dipole to be (their Eq. (34)),

$$\boldsymbol{P}_{em} = -\frac{1}{2}(\boldsymbol{p} \times \boldsymbol{B}), \tag{2}$$

where $\boldsymbol{p}$ is the dipole moment of the electric dipole and $\boldsymbol{B}$ is the (constant and uniform) magnetic field inside the shell. They then use a formula they propose for the calculation of hidden momentum to show that the hidden momentum is the negative of that in the above equation, resulting in zero total momentum. The formula they use, however, is just the negative of the actual electromagnetic momentum and will always trivially cancel it.

The real problem is that the process by which such a system is assembled is ignored by the authors. For example, you might start by bringing in electric charges from "infinity" to form the uniformly charged shell. The exterior forces necessary to do this sum to zero, so no momentum is added to the system. At the center of the shell is an object from which a dipole may be made. Once again, the external forces that create the necessary charge separation and hold the shell and dipole stationary are equal and opposite. Finally, a force couple is employed to start the charged shell rotating about an axis that is not coaxial with the dipole moment. Now you have a problem, as the dipole experiences a force due to the increasing magnetic field, yet the system has been considered to be at rest for the purposes of computation.

Consider the effect of imparting rotation to their shell of charge centered on a finite dipole moment consisting of charges q and $-q$ separated by a distance $2a$. For purposes of illustration, it is sufficient to examine the special case where the dipole moment points in the y direction while the magnetic field increases from 0 to B in the z direction. The two dipolar charges, separated by the distance $2a$, define the diameter of a circle of area πa^2, whose area is perpendicular to the magnetic field. The electric field due to the *emf* generated around this circle is $(1/2)aB/t$, where the field increases from zero to maximum in a time t. This results in a momentum impulse of $(1/2)aB/t \times 2qt = (1/2)(2aq)B = (1/2)pB$ in the positive x direction, as the point dipole moment, p, is formed by the usual limiting process. (The product qa is held constant as q goes to infinity and a goes to zero. The impulse remains constant during this process.) Generalizing to an arbitrary orientation of the electric dipole, the impulse is $(1/2)\boldsymbol{p} \times \boldsymbol{B}$.

Compared to Eqs. (34) and (49)of Babson *et al*, it is clear that, for their models, the total momentum, electromagnetic plus mechanical, remains zero. The system recoils in the positive x direction with a momentum that is equal and opposite to that contained in the electromagnetic field. Thus the problem with the calculations of Babson *et al*[3] is that they ignore the momentum that appears due to the assembly of the system.

III. A SPHERE WITH A DIPOLAR SURFACE CHARGE AND A UNIFORM SURFACE CURRENT DENSITY

This model is discussed in[5] and consists in that reference of a uniform sphere of radius R with a surface charge distribution given by

$$\sigma(\theta, \phi) = k sin\theta sin\phi, \tag{3}$$

where k is a constant and θ and ϕ are the polar and azimuth angles of the spherical coordinate system centered on the sphere. The the electric dipole moment is given by[6]

$$\boldsymbol{p} = \frac{4\pi}{3} R^3 k \hat{\boldsymbol{j}}. \tag{4}$$

To complete the model a thin spherical shell with a uniform surface charge density σ_o, rotating with angular velocity $\boldsymbol{\omega}$, encloses the sphere, is concentric with it, and has essentially the same radius. The rotating charge results in a uniform surface current density given by

$$\begin{aligned} \boldsymbol{K} &= \sigma_o \boldsymbol{\omega} \times \boldsymbol{R} = \sigma_o \omega R \hat{\boldsymbol{k}} \times \hat{\boldsymbol{r}} \\ &= \sigma_o \omega R(-sin\theta sin\phi \hat{\boldsymbol{\imath}} + cos\theta sin\phi \hat{\boldsymbol{\jmath}}). \end{aligned} \tag{5}$$

This surface current produces a uniform magnetic field inside the shell[3],

$$\boldsymbol{B}_o = \frac{2}{3}\mu_o \sigma_o R \omega \hat{\boldsymbol{k}}, \tag{6}$$

and a dipolar magnetic field outside the shell[3],

$$\boldsymbol{B} = \frac{\mu_o}{4\pi r^3}[3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}], \tag{7}$$

with the magnetic moment given by

$$\boldsymbol{m} = \frac{4\pi}{3}\sigma_o R^4 \omega \hat{\boldsymbol{k}}. \tag{8}$$

This model is quite similar to the one of Babson *et al*[3], but with point dipole replaced by a charge distribution with dipolar symmetry. As in that model, the assembly of the system and consequent momentum, should not be ignored. In this calculation the charged shell will initially be at rest and contain a thin dielectric shell with essentially the same radius and a dipolar surface charge density given by Eq.(3). Equal and opposite external forces hold this configuration stationary. The dielectric shell will have an electric field in its interior given by

$$\boldsymbol{E}_o = \frac{1}{4\pi\epsilon_o}\frac{\boldsymbol{p}}{R^3}. \tag{9}$$

Outside this shell the electric field will be that of a dipole,

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\frac{1}{r^3}[3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}]. \tag{10}$$

There will also be an electric field outside of R due to the shell of uniform charge density like that of a point charge at the center of the concentric shells. Once the uniformly charged shell is rotating, it will produce a dipolar magnetic field outside R, but, since the contribution to the electromagnetic momentum of this electric field and the dipolar magnetic field is zero, these fields will be ignored.

Now have the outer, uniformly charged shell begin rotating, uniformly increasing its angular speed from zero to ω. The angular acceleration will be small enough that radiation effects can be ignored. The increasing rotation will produce a uniformly increasing magnetic field inside the shell given by

$$\boldsymbol{B}_o(t) = \frac{2}{3}\mu_o\sigma_o R\dot{\omega}t\hat{\boldsymbol{k}}, \tag{11}$$

where $\dot{\omega}$ is the constant angular acceleration. The angular acceleration will continue until the magnetic field reaches the value in Eq. (6).

The increasing magnetic field inside the shells will produce an emf there. By Faraday's law this emf will be

$$\mathcal{E} = -\frac{d\Phi}{dt}, \tag{12}$$

where Φ is the magnetic flux $\pi R^2 sin^2\theta$. The emf is directed in the negative $\hat{\boldsymbol{\phi}}$ direction by Lenz' law, resulting in a Faraday electric field given by

$$\boldsymbol{E} = -\frac{\mathcal{E}}{2\pi R sin\theta}\hat{\boldsymbol{\phi}} = -\frac{1}{2}R sin\theta \dot{B}_o(-sin\phi\hat{\boldsymbol{\imath}} + cos\phi\hat{\boldsymbol{\jmath}}), \tag{13}$$

where $\dot{B}_o$ is the time rate of change of B_o.

This electric field will exert a force on an element of charge

$$dq = \sigma dA = (k sin\theta sin\phi)(R^2 sin\theta d\theta d\phi) = kR^2 sin^2\theta sin\phi d\theta d\phi, \tag{14}$$

in the dipolar shell of

$$d\boldsymbol{F} = \boldsymbol{E}dq = -\frac{1}{2}R sin\theta \dot{B}_o(-sin\phi\hat{\boldsymbol{\imath}} + cos\phi\hat{\boldsymbol{\jmath}})(kR^2 sin^2\theta sin\phi d\theta d\phi). \tag{15}$$

The integral of $sin\phi cos\phi$ over ϕ eliminates the $\hat{\boldsymbol{\jmath}}$ term. The integral over ϕ of $sin^2\phi$ yields

$$d\boldsymbol{F} = \frac{1}{2}\pi R^3 \dot{B}_o sin^3\theta d\theta \hat{\boldsymbol{\imath}}, \tag{16}$$

then the integral over θ gives

$$\boldsymbol{F} = \frac{2}{3}\pi R^3 \dot{B}_o \hat{\boldsymbol{\imath}}. \tag{17}$$

Using the value for the electric dipole $\boldsymbol{p}$ in Eq. (4), the above equation becomes

$$\boldsymbol{F} = \frac{1}{2}p\dot{B}_o\hat{\boldsymbol{\imath}}, \tag{18}$$

and since the force is the time rate of change of momentum, we finally have the impulse applied to the dipolar shell as a result of the creation of the magnetic field,

$$\boldsymbol{P} = \frac{1}{2}pB_o\hat{\boldsymbol{\imath}}. \tag{19}$$

This result obviously implies that, for an arbitrary orientation of the dipoles,

$$\boldsymbol{P} = \frac{1}{2}\boldsymbol{p} \times \boldsymbol{B}_o. \tag{20}$$

The next job is to calculate the electromagnetic linear momentum stored in the electromagnetic field. There are two contributions: one due to the fields inside the shells and one due to the dipolar fields outside the shell. Inside the shell Eq. (1) is just

$$\boldsymbol{P}_{in} = \epsilon_o E_o(-\hat{\boldsymbol{\jmath}}) \times B_o\hat{\boldsymbol{k}}(\frac{4}{3}\pi R^3) = -\frac{1}{3}pB_o\hat{\boldsymbol{\imath}} \quad \rightarrow \quad -\frac{1}{3}\boldsymbol{p} \times \boldsymbol{B}_o. \tag{21}$$

The calculation of the electromagnetic momentum outside the shell is straight-forward and was done by Babson *et al*[3], their Eq. (33b). Their result is

$$\boldsymbol{P}_{out} = -\frac{1}{6}\boldsymbol{p} \times \boldsymbol{B}_o. \tag{22}$$

Summing up Eqs. (20), (21), and (22) to get the total momentum, you see that it is zero without the necessity of appealing to hidden momentum.

IV. LINEAR MOMENTUM OF A POINT CHARGE IN THE VICINITY OF A MAGNETIC DIPOLE

In this section I revisit the calculation by Furry[4] of the linear electromagnetic momentum of a point charge in the vicinity of a magnetic dipole. The basic setup is indicated in Fig. 3. However, instead of having a magnet that is unpolarized, I will replace Furry's magnet

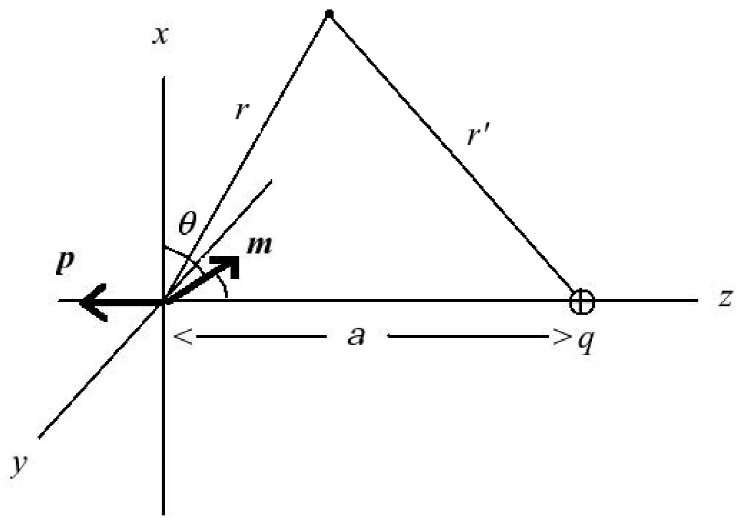

FIG. 1. A charge q in the vicinity of a magnetic moment $\boldsymbol{m}$ and the induced dipole moment $\boldsymbol{p}$.

with a conducting spherical shell of radius b that can carry a current and be polarized by an external charge. (I treated the case of a non-polarizable magnet here[7]). The system is assembled by having a positive point charge q move from a great distance along the x axis in the positive x direction to a position $x = -a$. It will polarize the small conducting shell located at the origin distance a from the point charge with an electric dipole given by Eq. (4), which will produce a uniform electric inside the shell given by Eq. (9). (In order to be more in concert with Furry's notation, I've replaced R with b.)

Once the point is in place, an external agent creates a uniform surface current in the conducting sphere. This will produce a magnetic field inside the sphere given by Eq. (6), and a magnetic dipole given by Eq. (8). The changing magnetic field at the location of the point charge will produce an impulse on the charge given by[2]

$$\Delta P = \frac{1}{2}\boldsymbol{p} \times \boldsymbol{B}_o. \tag{23}$$

(This is one-half the amount given in the above reference due to the presence of one charge here instead of two.) Another impulse is imparted to the shell due to the changing magnetic flux inside the shell. This is given byEq. (19). This is of the same magnitude and direction as the impulse to the charge, hence the total impulse received by the system during assembly is $\boldsymbol{p}\times\boldsymbol{B}_o$. To keep the charge stationary, an external agent must exert a force on it, receiving an impulse equal to this. Now the question is whether or not the electromagnetic momentum in the charge-magnet system is equal is equal and opposite, as it has to be for the total momentum to remain zero.

There are three contributions to the electromagnetic linear momentum: that due to the field of the external charge and dipolar magnetic field outside the shell, that due to the uniform fields inside the shell, and that due to the external dipolar magnetic field with the external dipolar electric field of the shell. All these have previously been calculated. Respectively, they are, $-(1/2)\boldsymbol{p} \times \boldsymbol{B}_o$[4]; $-(1/3)\boldsymbol{p} \times \boldsymbol{B}_o$, Eq. (21); and $-(1/6)\boldsymbol{p} \times \boldsymbol{B}_o$, Eq. (22). These add up to $-\boldsymbol{p} \times \boldsymbol{B}_o$, and momentum is conserved.

V. A ROTATING MAGNETIC DIPOLE IN A UNIFORM ELECTROSTATIC FIELD

This example was explored by Mansuripur[8]. He considers a slowly rotating magnetic dipole at the origin of a Cartesian coordinate system, rotating in the x-y plane:

$$\boldsymbol{m}(t) = m_o[cos(\omega t)\hat{\boldsymbol{\imath}} + sin(\omega t)\hat{\boldsymbol{\jmath}}], \tag{24}$$

where ω is the angular speed of rotation and m_o is the magnitude of the dipole moment. The dipole is positioned between two infinite parallel non-conducting plates, each parallel to the y-z plane, and each a distance $d/2$ from $\boldsymbol{m}$ along the y axis. Hence the plates are a distance d apart with the rotating dipole halfway between them. The plate at $x = -d/2$ has a uniform surface charge of σ, and the plate at $x = d/2$ has a uniform surface charge $-\sigma$. The uniform electric field between the plates is given by

$$\boldsymbol{E} = \frac{\sigma}{\epsilon_o}\hat{\boldsymbol{\imath}}. \tag{25}$$

Mansuripur calculates the oscillating force acting on the plates due to the induced electric field resulting from the time rate of change of the magnetic field of the dipole. His result is

$$\boldsymbol{F}(t) = -\sigma m_o \omega cos(\omega t)\hat{\boldsymbol{k}}. \tag{26}$$

He also calculates the oscillating force on the magnetic dipole in the Einstein-Laub[9] formulation and shows that it is equal and opposite to the force on the plates, conserving linear momentum.

Next, Mansuripur addresses the same system in the Lorentz formulation. He notes that in this formulation there is no force on the dipole due to the electric field, in contradiction to that found for Einstein-Laub. However, he notes there is an oscillating electromagnetic linear momentum in the electromagnetic fields given by

$$\boldsymbol{p}_{em} = \epsilon_o \boldsymbol{E} \times \boldsymbol{m} = \sigma m_o sin(\omega t)\hat{\boldsymbol{k}}. \tag{27}$$

The time rate of change of this momentum would appear to account for the force in Eq. (26) satisfying momentum conservation, but he rejects this interpretation, citing references that claim there is hidden momentum in magnets residing in electric fields. However, I have shown that there is no hidden momentum in a magnet in an electric field[7]. To accomodate the idea of hidden momentum, he has to invent a force acting on the dipole in the Lorentz formulation, although the source of this force is obscure. It is simpler to just conclude the time changing momentum in the electromagnetic fields cancels the force on the plates such that no hidden momentum is invoked.

I haven't yet addressed the assembly of this system. In previous examples it has been shown that the assembly is key to understanding why hidden momentum does not exist in these examples. However, for this example there is a slowly rotating magnetic dipole creating a quasi-static situation. Let the dipole be originally non-rotating and then apply a gentle torque to gradually spin it up. If the spin-up is slow enough the quasi-sinusoidal nature of the motion will largely lead to cancelation of momentum and forces in the system when averaged over the time involved to bring the dipole to its final slow rotation. Hence assembly of the system is negligible if the spin-up occurs over a sufficiently long time.

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