On the fields of moving dipoles

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Note: This is a work in progress. An observer moving parallel to a current-carrying wire detects an electric field due to the Lorentz transformation directed either toward or away from the wire, depending on the relative motion of observer and current. The accepted interpretation of this situation as viewed from the observer's rest frame is that there is a net linear charge density on the wire: The Lorentz contraction of the separation of fixed ions and charge carriers is different due to their different speeds in the observer's frame. The idea that a net charge exists on a wire in a reference frame moving parallel to the wire is compatible with the Lorentz-transformed fields for a straight wire but fails to give the correct transformed electric field for a current loop. I argue that the apparent charge on a current-carrying wire is due to the relativity of simultaneity. The phenomenon of Lorentz contraction is a consequence of this such that there is no charge separation on a moving current loop. This has implications concerning the interpretation of the nature of the induced electric dipole that appears on a moving magnetic dipole.

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I. INTRODUCTION

The presence of a radial electric field centered on a current-carrying wire detected by observers moving parallel to the wire has been thought to be due to a net linear charge density in the wire as seen by the observers. This charge density is supposed to be a result of the difference in the separation between positive charges versus negative charges in the wire due to Lorentz contraction, since the charges in the wire are observed to be traveling at different speeds [1, 2]. In the reference frame at rest with the wire, the charge carriers are in motion and the fixed ions are stationary. In a frame moving parallel to the wire both types of charges are in motion, but one is moving faster than the other. The idea is that an observer moving parallel to the wire (or with a component of velocity parallel to the wire) will see a difference in the Lorentz contraction of the average separation of the two types of charges, with the faster charges appearing closer together than the slower ones.

This effect is what I call "differential Lorentz contraction". According to this idea, the difference in average separation results in a net linear charge density on the wire in the moving reference frame, which produces the radial electric field felt by a charge moving parallel to the wire. Hence, a readily observable phenomenon, for example, the force of attraction or repulsion acting between parallel wires carrying currents, is thought to be a direct manifestation of length contraction, which would otherwise might not be seen except in objects traveling near the speed of light.

The Lorentz transformation of the wire's magnetic field from its rest frame to a frame moving parallel to the wire results in a cylindrically symmetric radial electric field centered on the wire. The electric field lines either originate on the wire and extend radially outward or converge inward and terminate on the wire, implying a net charge density on the wire in the moving frame. This conclusion appears to be based on a misleading application of Gauss' law, as it ignores the timing as perceived by observers moving parallel to the wire.

The idea that the radial electric field lines resulting from a Lorentz transformation of the wire's magnetic field is due to a net charge on the wire due to differential Lorentz contraction runs into difficulty when applied to current loops. For a moving current loop it is supposed that one side of the loop is positive and the other negative. In that case the electric field lines should all originate on the positive side of the loop and teminate on the negative side. However, this is not true of the electric field lines of the Lorentz-transformed magnetic field of the loop. For example, the transformed field lines in the plane of the loop are all perpendicular to the direction of the loop's motion and extend from plus to minus infinity. This would imply that there is charge at an infinite distance from the loop were these lines due to an electrostatic field.

There are two ways an electric field can be generated. An electrostatic field originates on positive charges and terminates on negative charges and is due to Gauss' law. A Faraday field arises from a magnetic field changing with time (Faraday's law) and neither originates nor terminates on electrical charges. In addition, how an electromagnetic field is perceived by an observer depends on the observer's velocity. The idea that the electrical field of a moving magnetic dipole involves a charge separation on the dipole is incompatible with the actual field lines. I will discuss this in greater detail later on when these fields are derived.

The appearance of an electric dipole on a moving magnetic dipole (or a magnetic dipole on a moving electrical dipole) is inferred from applying the Lorentz transformation to the magnetization-polarization tensor [3]. However, as was pointed out above and is well known, calculating the transformed electromagnetic fields assuming they are due to these induced dipoles does not give the same fields that result from a Lorentz transformation of the fields of the moving dipoles themselves. This leads to questions as to what algebraic expression resulting from the multipole expansions of fields is "the" induced dipole [6, 7]. In this paper I address these problems and argue that you cannot Lorentz-transform dipoles from one reference frame to another as you can fields. In essence, there is no "the" dipole.

II. DIFFERENTIAL LORENTZ CONTRACTION VERSUS TRANSFORMED FIELDS

The theoretical basis for differential Lorentz contraction was given most clearly, in my opinion, by Gabuzda [1]. In this paper she argued that a current carrying wire between a source and sink of charge was neutral in the rest frame of the wire despite the expected increase in linear electron density due to Lorentz contraction of the average electron spacing. An observer stationary with respect to the wire would detect that the electrons were closer together due to their drift speed if this were the only effect on the electron density.

However, the source and sink of charge are also in the rest frame of the wire and would see the wire as charged. Gabuzda argued that the excess electrons would move into the electron sink, making the wire neutral in its rest frame. Then, transforming the system of wire plus source and sink to the frame of an observer moving with speed v parallel to the wire, she showed that the fields seen by the observer would be the same as those calculated from a Lorentz transformation of the fields from the rest frame of the wire to that of the observer.

As for current loops without source or sink, such as created by changing magnetic flux, Gabuzda notes that the loop as a whole will remain neutral in spite of Lorentz contraction and so a test charge at rest with the loop would not experience a force. She does not consider the case where the loop and charge are in relative motion. Unless the charge is along the symmetry axis of the loop and the loop is moving along that axis, there will be an electric field present at the position of the charge due to the Lorentz-transformed magnetic field of the loop to the charge's frame of reference. The differential Lorentz contraction point of view is that one side of the loop is positively charged and the other side negative in the stationary frame. This view led Mansuripur [4] to propose the paradox named after him.

In the Mansuripur paradox there is a charge in the vicinity of a current loop and at rest in the loop's reference frame. To simplify the problem the charge is taken to be in the plane of (but outside) the loop. One can imagine the charge is at the origin of a Cartesian coordinate system and the magnetic moment of the current loop is on the positive x axis with the dipole moment oriented in the positive z direction. The Lorentz transformation of a stationary magnetic dipole to a moving frame of reference is thought to result in an electric dipole centered on the magnetic dipole. In the frame of reference of an observer moving toward the charge-loop system in the positive x direction, the electric dipole moment is oriented perpendicular to the magnetic dipole in the positive y direction.

According to the idea of differential Lorentz contraction, the moving observer sees the left side of the loop positive (in the positive y direction) and the right side negative. Hence it is supposed he must see that the loop experiences a torque due to the action of the field of the point charge on the charge separation, a torque not seen by an at-rest observer. These two different observations are obviously contradictory. Mansuripur [4] claimed the contradiction can be resolved if the Lorentz force responsible for the torque were replaced by a force proposed by Einstein and Laub [5].

There are significant problems with the idea there is a charge separation on the current loop. An electric dipole due to a charge separation as described above would produce an electric field in the x-y plane that would consist of loops originating on the side of the loop with the positive charge and terminating on the side of the loop with the negative charge. An observer moving as indicated above should see an electric field pointing in the negative y direction if this were the case.

However, applying the right-hand rule to her situation, it is clear she sees the electric field in the *positive y* direction. In fact the electric field in the x-y plane is in the positive y direction *everywhere* outside the loop! In addition there is no electric field parallel to the direction of motion of the loop, as will be shown below. This is not the field of an electric dipole due to a charge separation. It is an error to consider the electric dipole that results from a Lorentz transformation of a magnetic dipole to be the same as that of an electric dipole with an actual charge separation. This has been noted in the literature and work has been done to explain the fields of moving magnetic and electric dipoles. For example, see [6, 7].

The resolution of the Mansuripur paradox has been thought by those supporting the Lorentz force to be due to the presence of hidden momentum in the charge-magnetic loop system [8–12]. I have shown [13] that the actual resolution has to do with the transformation of the linear and angular electromagnetic momentum in the charge-magnetic loop system [14] from its rest frame to the lab frame in which it is moving. The torque that results from this transformation cancels that due to the charge-"electric dipole" interaction. This resolution also preserves the Lorentz force.

The torque that is supposed to be acting on the loop due to the presence of the point charge is not due to the interaction between the point charge and a charge separation on the loop. Instead, this torque exists as a time component of the torque four-vector in the rest frame of the charge-loop system. The charge-loop system also contains linear and angular electromagnetic momentum in the electromagnetic field [14]. The linear momentum is expressed in a time component of the angular momentum four-tensor. When these four-tensors are Lorentz transformed to the lab frame, the torques that appear as space components cancel [13]. No hidden momentum is necessary, therefore, to resolve this paradox. (In reference [13] I referred to the torques as "fictitious". This was a poor choice of words as the torques are not fictitious but originate in the electromagnetic field.)

III. LORENTZ TRANSFORMATION OF THE CURRENT DENSITY OF A WIRE

A fully relativistic treatment of the fields around a current-carrying wire involves the Lorentz transformation of Maxwell's equations in covariant form. The field-strength tensor [15], which is the covariant form of the electric and magnetic fields, is given (in SI units) by

$$(F^{\mu,\nu}) = \begin{pmatrix} 0 & B_3 & -B_2 & -i\dfrac{E_1}{c} \\ -B_3 & 0 & B_1 & -i\dfrac{E_2}{c} \\ B_2 & -B_1 & 0 & -i\dfrac{E_3}{c} \\ i\dfrac{E_1}{c} & i\dfrac{E_2}{c} & i\dfrac{E_3}{c} & 0 \end{pmatrix} \tag{1}$$

where the coordinates are x_1, x_2, x_3, $x_4 = ict$; i equals the square root of -1; and where the subscript 4 corresponds to the time coordinate [16].

Let there be a current density J' in a primed frame moving in the lab (unprimed) in the x_3 direction. (This will later become the current I' in a wire lying along the x_3 axis and moving parallel to it.) The wire is neutrally charged so that the electric field in the primed frame is zero. Also there is no magnetic field in the x_3 direction. The field-strength tensor becomes

$$(F^{\mu',\nu'}) = \begin{pmatrix} 0 & 0 & -B_2' & 0 \\ 0 & 0 & B_1' & 0 \\ B_2' & -B_1' & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2}$$

The inhomogeneous Maxwell equations are written in covariant form [17] as

$$\sum_{\nu'} \frac{\partial F^{\mu'\nu'}}{\partial x^{\nu'}} = \mu_o J^{\mu'}, \tag{3}$$

in SI units, where

$$J^{\mu'} = (\boldsymbol{J}', i\rho' c) = (J_1', J_2', J_3', i\rho' c), \tag{4}$$

in general, and where $\boldsymbol{J}'$ is the current density and ρ' is the charge density, and, for this case,

$$J^{\mu'} = (0, 0, -J', 0). \tag{5}$$

(The minus sign if front of J' is needed so that the moving observers in this derivation are traveling opposite to the positive current, as will be assumed in the relativity-of-simultaneity derivation of the next section.)

To transform these Maxwell equations from the rest (unprimed) frame a moving (primed) frame parallel to the wire (in the x_3 direction), you must apply the Lorentz transformation [18] to Eq. (3). In matrix form you have

$$\left(\sum_{\nu} \frac{\partial F^{\mu\nu}}{\partial x_\nu}\right) = \begin{pmatrix} \dfrac{\partial B_3}{\partial x_2} - \dfrac{\partial B_2}{\partial x_3} - \dfrac{1}{c^2}\dfrac{\partial E_1}{\partial t} \\ -\dfrac{\partial B_3}{\partial x_1} + \dfrac{\partial B_1}{\partial x_3} - \dfrac{1}{c^2}\dfrac{\partial E_2}{\partial t} \\ \dfrac{\partial B_2}{\partial x_1} - \dfrac{\partial B_1}{\partial x_2} - \dfrac{1}{c^2}\dfrac{\partial E_3}{\partial t} \\ \dfrac{i}{c}\dfrac{\partial E_1}{\partial x_1} + \dfrac{i}{c}\dfrac{\partial E_2}{\partial x_2} + \dfrac{i}{c}\dfrac{\partial E_3}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma_v & i\gamma_v\beta \\ 0 & 1 & -i\gamma_v\beta & \gamma_v \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\mu_o J' \\ 0 \end{pmatrix} \tag{6}$$

where $\beta = v/c$, and the individual equations of (6) are the inhomogeneous Maxwell equations in the lab frame. They are

$$\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3} - \frac{1}{c^2}\frac{\partial E_1}{\partial t} = 0 \tag{7}$$

$$-\frac{\partial B_3}{\partial x_1}+\frac{\partial B_1}{\partial x_3}-\frac{1}{c^2}\frac{\partial E_2}{\partial t}=0 \tag{8}$$

$$\frac{\partial B_2}{\partial x_1}-\frac{\partial B_1}{\partial x_2}-\frac{1}{c^2}\frac{\partial E_3}{\partial t}=-\gamma_v\mu_o J' \tag{9}$$

$$\frac{1}{c}\left(\frac{\partial E_1}{\partial x_1}+\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right)=\frac{v}{c}\gamma_v\mu_o J' \tag{10}$$

In what may be more familiar form, these equations are

$$\boldsymbol{\nabla}\times\boldsymbol{B}-\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}=-\gamma_v\mu_o\boldsymbol{J'}=-\mu_o\boldsymbol{J}, \tag{11}$$

and

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=v\gamma_v\mu_o J'=\frac{\rho}{\epsilon_o}. \tag{12}$$

The current density is due to a narrow flow of charge along the x_3 axis. It can be expressed in Dirac delta functions as

$$J'=I'\delta(x_1')\delta(x_2')=\gamma_{u'}en'u'\delta(x_1)\delta(x_2), \tag{13}$$

where u' is the drift speed in the moving frame,

$$I'=\gamma_{u'}en'u', \tag{14}$$

and where $\gamma_{u'}$, is given by

$$\gamma_{u'}=\frac{1}{\sqrt{1-\dfrac{u'^2}{c^2}}}. \tag{15}$$

Also used is the fact that, for this Lorentz transformation, $x_1=x_1'$ and $x_2=x_2'$.

Equation (12) is Gauss' law. The right-hand side of this equation multiplied by ϵ_o is the charge density ρ in the lab frame. Is this a real charge density? It certainly appears to be as it makes the divergence of the Lorentz-transformed electric field nonzero. However, it is proportional to a flow of charge. Look at the conservation equation for electrical charge:

$$\nabla\cdot\boldsymbol{J}+\frac{\partial\rho}{\partial t}=0. \tag{16}$$

The divergence of $\boldsymbol{J}$ is non-zero in a reference frame where the charge density at a given location is time-dependent, as is the case here [7], but this does not necessarily imply that there is a charge separation. This will be discussed further in the next section.

To get an expression for the differential "charge density" on the wire, you integrate ρ over x_1 and x_2. The result is (using $\mu_o\epsilon_o=1/c^2$)

$$\begin{aligned} e\Delta n &= \int \rho\, dx_1 dx_2=\frac{v}{c^2}\gamma_v I'\int \delta(x_1)\delta(x_2)dx_1dx_2 \\ &= \frac{v}{c^2}\gamma_v I', \end{aligned} \tag{17}$$

or, using Eq. (14),

$$e\Delta n=\gamma_v\gamma_{u'}\frac{u'v}{c^2}en' \tag{18}$$

To finish the solution, you solve the Maxwell equations, noting that, by symmetry, B_3' and E_3' are zero as are derivatives with respect to x_3'. The well-known results are that the magnitude of the magnetic field turns out to be

$$B=\gamma_v\frac{\mu_o}{2\pi}\frac{I'}{r}, \tag{19}$$

with the electric field magnitude given by $E=vB$.

IV. THE APPEARANCE OF CHARGE ON A LOOP IS DUE TO THE RELATIVITY OF SIMULTANEITY

In this section I will argue that the apparent linear charge, Eq.(18), on a current-carrying wire is due to the relativity of simultaneity, not to a physical charge density. In Fig. 1 you see the (vertical) worldlines of the ends (A' and B') of a segment of a current-carrying wire in the spacetime diagram of the wire's rest frame, where the current is flowing in the negative x direction. These are also the worldlines of two observers on the wire. Additionally there are the world lines of two stationary observers (the diagonal lines A and B as seen from the wire's frame) as the wire moves past their positions in the negative x direction. The line at A is also the ct axis of the stationary observers spacetime diagram – the unprimed frame). The wire segment has a proper length of l, whereas the two end stationary observers are γl apart in their rest frame and l apart in the primed frame. The two dots on the horizontal line are events involving two charge carriers a distance l apart in the primed frame, where it has been arranged that the two stationary observers are at the locations of the two observers moving with the wire when the charge carriers pass the positions of the two observers on the wire. These two spacetime events occur at the same time in the moving (wire) frame.

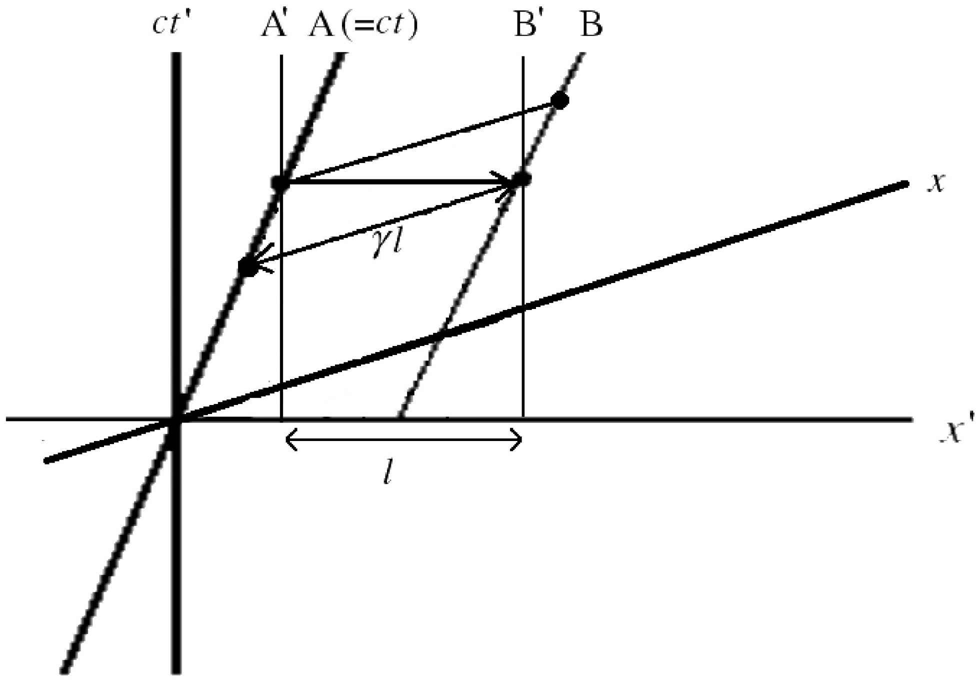

FIG. 1. Spacetime diagrams for at rest and moving observers along an at rest current-carrying wire. See text for explanation.

Now if you only consider Lorentz contraction, you take the distances between the charges to be l as seen in the moving frame. However, in the stationary frame the two events defining separation of the charges are not simultaneous like they are in the moving frame. When the right stationary observer B reaches moving observer B$'$, left observer A has yet to reach observer A$'$. By the time observer A does reach A$'$, observer B is farther down the wire and a certain amount of time has elapsed since she was abreast of B$'$. In the moving time frame, B considers an amount of charge has traveled down the wire that observer A has yet to see. Hence there is an excess amount of charge on the wire in their reference frame at a given unprimed time t..

To quantify this, imagine a rectangular current loop with its length l' parallel to the x axis and moving in the negative x direction (to the left) with speed v in the lab (unprimed) frame (Fig. 2). A positive current I' is circulating in the rest frame of the loop in a counterclockwise direction such that it is flowing to the left in the upper side of the loop. There are n' positive charge carriers per unit length in the loop, each with a charge e (making the fixed ions negative). The average drift speed of the charge carriers is u' in the loop's rest frame, such that an observer moving along with the loop sees a positive current given by Eq. (14).

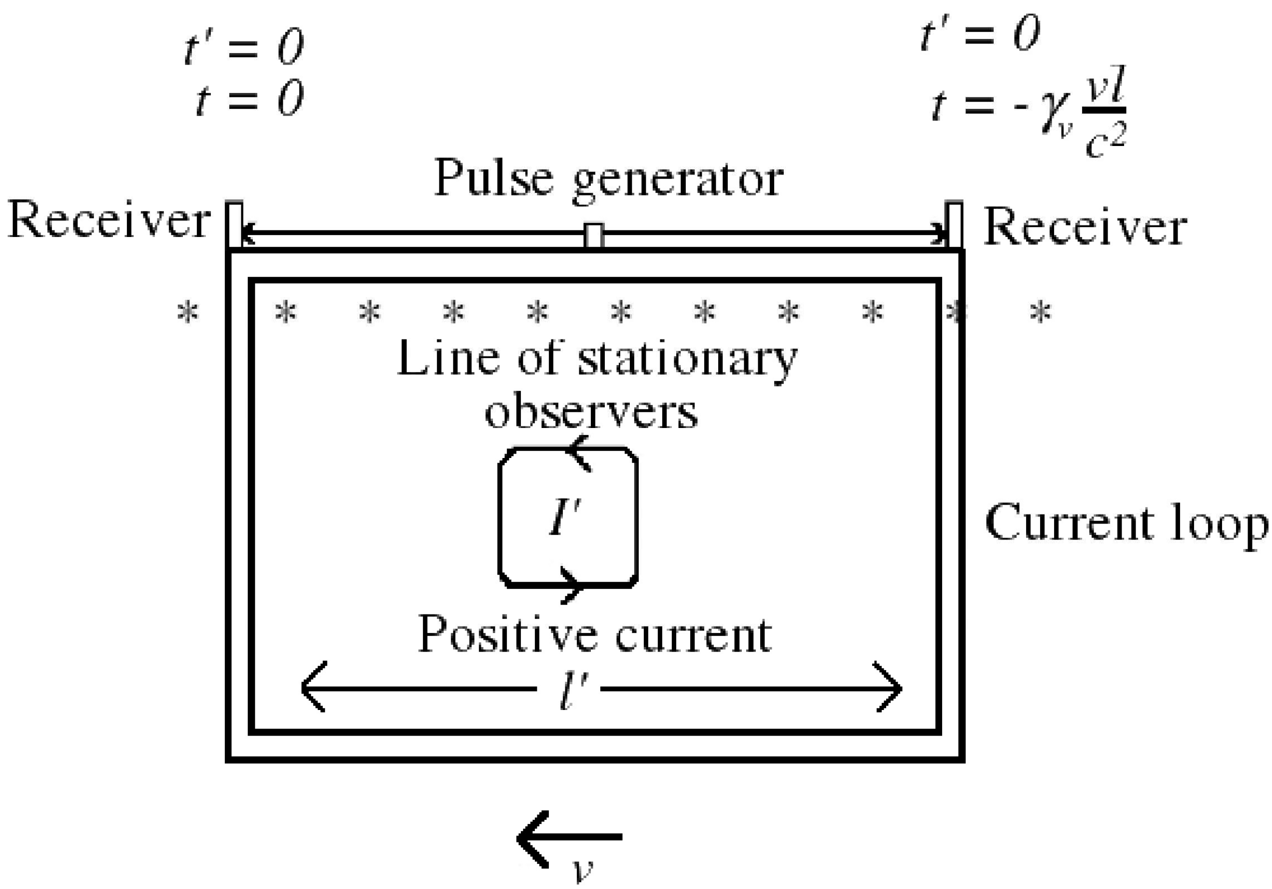

FIG. 2. What a observer (at a sufficient distance) would see when moving at the same speed v as the apparatus described in the text. (The primed values refer to the "at-rest" frame.)

In the lab frame the drift speed due to relativistic velocity addition [19], would be

$$u = \frac{u' + v}{1 + \dfrac{u'v}{c^2}} - v = \frac{(c^2 - v^2)u'}{c^2 + u'v}, \tag{20}$$

and a current I given by

$$I = enu = e(n'\gamma_{u+v})\frac{(c^2 - v^2)u'}{c^2 + u'v} = \frac{\gamma_{u'}}{\gamma_v} en'u', \tag{21}$$

where

$$n = n'\gamma_{u+v} = n'\frac{1}{\sqrt{1 - \dfrac{(u+v)^2}{c^2}}} = n'\frac{c^2 + u'v}{\sqrt{(c^2 - v^2)(c^2 - u'^2)}} = n'\gamma_{u'}\gamma_v\frac{c^2 + u'v}{c^2}. \tag{22}$$

Now imagine that there are observers in the lab frame with synchronized clocks, lined up to observe the current in the upper wire of the loop as it passes each of their positions. Borrowing an example of the relativity of simultaneity from page 38 of Wolfgang Rindler's book, *Relativity: Special, General, and Cosmological* [20], imagine that there is a laser pulse generator located halfway along the wire which projects identical pulses of sufficient briefness at the same time toward two receivers at each end of the wire. A stationary observer will see a pulse approach the leading receiver on the left at a relative speed of $c-v$ and the trailing receiver on the right at a relative speed of $c+v$. The distance the pulses travel will be the Lorentz-contracted distance $l'/2\gamma_v$. Hence, the the observer who is abreast of the trailing receiver when the pulse arrives will record a time of arrival of

$$t = \frac{l'}{2\gamma_v(c-v)} - \frac{l'}{2\gamma_v(c+v)} = \gamma_v \frac{vl'}{c^2} \tag{23}$$

earlier than that of an observer at the leading receiver when the pulse arrives there. Therefore, by her estimation, an amount of charge given by

$$q = I(t-0) = \left(\frac{\gamma_{u'}}{\gamma_v} en'u'\right)\left(\gamma_v \frac{vl'}{c^2}\right) = \gamma_{u'} \frac{u'v}{c^2} en'l' \tag{24}$$

has flowed past her position before the observer at the leading receiver sees the arrival of the pulse, using Eq. (21) for I'. That is, the observer at the leading plate has not seen this charge pass his position when the pulse arrives.

The stationary observers must conclude, upon comparing notes, that there was an excess charge of q on the wire in their reference frame equally distributed along the wire. They therefore expect there to be an excess linear charge density on the wire given by

$$e\Delta n = \frac{q}{l'/\gamma_v} = \gamma_v \gamma_{u'} \frac{u'v}{c^2} en' \tag{25}$$

in their reference frame, the same as that found in Eq. (18). On the other hand they realize that this extra charge was not present in the traveling frame and *was only seen in their frame due to the effect of the relativity of simultaneity.*

If this “charge density” is used to calculate an electric field, the result is

$$E = \frac{1}{2\pi\epsilon_o} \frac{\gamma_v \gamma_{u'} \frac{u'v}{c^2} en'}{r} = \gamma_v v B' \tag{26}$$

where B' is given by

$$B' = \frac{\mu_o}{2\pi} \frac{I'}{r}, \tag{27}$$

I' by Eq. (14), and $c^{-2} = \epsilon_o \mu_o$. Note that this result does not require an actual net charge to be present on the wire. When the observers compare notes, they see that the excess charge was the result of their measuring the charge flow at times that were not simultaneous with what was happening in the moving system, and this is apparently why they also infer what appears to be a radial electric field.

Eq. (25 is the same excess charge density as that found by Gabuzda [1] arguing from the point of differential Lorentz contraction. Differential Lorentz contraction occurs because of the relativity of simultaneity in the following way. Because of timing differences between the moving and stationary reference frames, the stationary observer at the trailing end of the current loop sees an excess charge move onto the upper wire of the loop that the leading observer does not see when they make observations at the same time in their reference frame. Hence, at identical times in their reference frame there are more charge carriers on the upper wire than there are at identical times in the moving frame, hence the charge carriers appear to be closer together. Yet, once again, it is the relativity of simultaneity that produces this effect.

Unfortunately, the observers' conclusion that the electric field of the moving wire is due to either the relativity of simultaneity or differential Lorentz contraction appears to be incorrect. In either the above case and in the case of an actual charge separation on the moving loop, the electric field due to the transformed magnetic field of the loop is not compatible with these scenarios.

Certainly, if there were an actual charge separation on the loop, there would have to be a dipolar electric field. However, there is no actual charge separation and the electric field around the loop is due to neither a charge separation on the loop nor the inferred charge resulting from the relativity of simultaneity. My interpretation of the situation is the Lorentz transformation a moving magnetic dipole moment produces an electric “dipole” that is actually the result of the inferred charge of the relativity of simultaneity, but this is not a real charge and therefore is not responsible for the electric field. Rather, I argue that the electric field around a moving current-carrying wire or a magnetic dipole due to the Lorentz transformation of the magnetic field does not depend on the source of the field. That is, the induced electric field does not depend on the existence of net charge concentrations.

Consider this thought experiment. Say you have a Gilbertian magnet consisting of a gap between two poles of magnetic charges creating a magnetic field $\boldsymbol{B}$ in the gap. An electric charge inserted into this gap with a velocity $\boldsymbol{v}$ would experience a Lorentz force. The Lorentz transformed electric field would be $\boldsymbol{v} \times \boldsymbol{B}$. However, there can be no charge separation or inferred charge due to the relativity of simultaneity here. The Lorentz transformation applied to electromagnetic fields is local. In my opinion the sources of the fields have nothing to do with this. Whatever the source of a magnetic field, be it Amperian or Gilbertian, the Lorentz transformation works the same way. In the next section I will show that the electric field due to the Lorentz transformation of the magnetic field of a dipole is not a dipole field.

V. THE TRANSFORMED FIELDS OF A MOVING MAGNETIC DIPOLE

The magnetic field of a point magnetic dipole located at the origin of a coordinate system in its rest (primed) frame is

$$\boldsymbol{B}' = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}'\cdot\boldsymbol{r}')\boldsymbol{r}'}{r'^5} - \frac{\boldsymbol{m}'}{r'^3}\right] - \frac{2\mu_o\boldsymbol{m}'}{3}\delta(\boldsymbol{r}'), \tag{28}$$

where $\boldsymbol{r}' = x'\boldsymbol{i} + y'\boldsymbol{j} + z'\boldsymbol{k}$ and $\delta(\boldsymbol{r}')$ is the Dirac delta function. This is also the magnetic field of a current loop (minus the Dirac delta function) at a distance from the loop large compared to the loop's radius. Transforming to a (unprimed) frame moving in the positive x direction with speed v, you find

$$\boldsymbol{B} = B_{x'}\boldsymbol{i} + \gamma B_{y'}\boldsymbol{j} + \gamma B_{z'}\boldsymbol{k} \tag{29}$$

and

$$\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B} = \gamma v B_{z'}\boldsymbol{j} - \gamma v B_{y'}\boldsymbol{k}. \tag{30}$$

The position vector in the primed frame transforms to $\boldsymbol{r}' = \gamma x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ so that $r'^2 = \gamma^2 r^2 - (\gamma^2-1)(y^2+z^2) = \gamma^2 r^2(1-(v^2/c^2)sin^2\alpha)$, where $\boldsymbol{r} = x\boldsymbol{k} + y\boldsymbol{j} + z\boldsymbol{k}$ is the position vector in the unprimed frame and α is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$.

The components of the magnetic field in the primed frame are

$$\begin{aligned} B_{x'} &= \frac{3\mu_o m'}{4\pi}\frac{x'z'}{r'^5}, \\ B_{y'} &= \frac{3\mu_o m'}{4\pi}\frac{y'z'}{r'^5}, \\ B_{z'} &= \frac{\mu_o m'}{4\pi}\left[\frac{3z'^2}{r'^5} - \frac{1}{r'^3}\right] - \frac{2\mu_o m'}{3}\delta(\boldsymbol{r}'). \end{aligned} \tag{31}$$

The transformed field components are (see [21] for the technique),

$$\begin{aligned} B_x &= \frac{\mu_o m'}{4\pi}\frac{3xz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \\ B_y &= \frac{\mu_o m'}{4\pi}\frac{3yz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \\ B_z &= \frac{\mu_o m'}{4\pi}\left[\frac{3z^2}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}} - \frac{1}{\gamma^2 r^3(1-(v^2/c^2)sin^2\alpha)^{3/2}}\right] - \frac{2\mu_o m'}{3}\delta(\boldsymbol{r}). \end{aligned} \tag{32}$$

and

$$\begin{aligned} E_x &= 0, \\ E_y &= \frac{\mu_o v m'}{4\pi}\left[\frac{3z^2}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}} - \frac{1}{\gamma^2 r^3(1-(v^2/c^2)sin^2\alpha)^{3/2}}\right] - \frac{2\mu_o v m'}{3}\delta(\boldsymbol{r}). \\ E_z &= -\frac{\mu_o v m'}{4\pi}\frac{3yz}{\gamma^4 r^5(1-(v^2/c^2)sin^2\alpha)^{5/2}}, \end{aligned} \tag{33}$$

Here, $\gamma\delta(\boldsymbol{r}') = \gamma\delta(x')\delta(y')\delta(z') = \gamma[(\delta(x)/\gamma)\delta(y)\delta(z)] = \delta(\boldsymbol{r})$ has been used.

Note that $(\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{r} = xz\boldsymbol{i} + yz\boldsymbol{j} + z^2\boldsymbol{k}$. This means you can express the equations in (32) as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}}{\gamma^4 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} - \frac{\boldsymbol{m}'}{\gamma^2 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right] - \frac{2\mu_o\boldsymbol{m}'}{3}\delta(\boldsymbol{r}). \tag{34}$$

Now $1-(v^2/c^2)sin^2\alpha = \gamma^{-2} + (v^2/c^2)cos^2\alpha$ where $cos\alpha$ is the direction cosine from the $\boldsymbol{v}$ direction. This allows the equation above to be written as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\gamma\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}}{r^5[1+\gamma^2(v^2/c^2)cos^2\alpha]^{5/2}} - \frac{\gamma\boldsymbol{m}'}{r^3[1+\gamma^2(v^2/c^2)cos^2\alpha]^{3/2}}\right] - \frac{2\mu_o\boldsymbol{m}'}{3}\delta(\boldsymbol{r}). \tag{35}$$

In the slow-motion case ($v << c$) you can set the quantities in the brackets in the denominators equal to one, such that the transformed dipole is $\boldsymbol{m} = \gamma\boldsymbol{m}'$. However, nothing appears to be gained by this since γ is taken to be one in the slow-motion case anyway.

The real problem is with the electric field resulting from the transformation, $\boldsymbol{E} = -\boldsymbol{v}\times\boldsymbol{B}$. Dropping the delta function, the electric field is

$$\boldsymbol{E} = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}\times\boldsymbol{v}}{\gamma^4 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} - \frac{\boldsymbol{m}'\times\boldsymbol{v}}{\gamma^2 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right]. \tag{36}$$

The usual definition of the electric dipole resulting from the motion of a magnetic dipole is $\boldsymbol{p} = \gamma\boldsymbol{v}\times\boldsymbol{m}'/c^2$. This quantity does not appear in the above equation in the first term inside the square brackets, meaning this cannot be the electric field of a dipole with an actual charge separation. In particular, there is no electric field parallel to the velocity of the magnetic dipole. However, the numerator of this term can be written as $(\boldsymbol{m}'\cdot\boldsymbol{r})(\boldsymbol{r}\times\boldsymbol{v}) = (\boldsymbol{v}\times\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r} - (\boldsymbol{v}\times\boldsymbol{m}')r^2 + (\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r})$. Substituting this and the expression for $\boldsymbol{p}$ in the above equation results in

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - 3\boldsymbol{p}r^2 + 3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{\gamma^5 r^5[1-(v^2/c^2)sin^2\alpha]^{5/2}} + \frac{\boldsymbol{p}}{\gamma^3 r^3[1-(v^2/c^2)sin^2\alpha]^{3/2}}\right]. \tag{37}$$

For the slow-motion case, this equation can be expressed as

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r} - 3\boldsymbol{p}r^2 + 3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5} + \frac{\boldsymbol{p}}{r^3}\right], \tag{38}$$

or

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\boldsymbol{p}}{r^3} + \frac{3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5} - \frac{\boldsymbol{p}}{r^3}\right]. \tag{39}$$

The first two terms in square brackets are those of a dipole field with a charge separation. Hence this equation can be written as

$$\boldsymbol{E} = \boldsymbol{E}_{dp} + \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{r}\times\boldsymbol{m}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5} - \frac{\boldsymbol{p}}{r^3}\right], \tag{40}$$

where $\boldsymbol{E}_{dp}$ is the dipole field.

VI. THE TRANSFORMED FIELDS OF A MOVING ELECTRIC DIPOLE

The electric field an electric charge q moving with velocity $\boldsymbol{v}$ is given by [21]

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\frac{q\boldsymbol{r}}{\gamma^2 r^3(1-\beta^2 sin^2\alpha)^{3/2}}, \tag{41}$$

where $\beta = v/c$ and the other symbols are defined as previously. The field due to two adjacent opposite charges separated by a vector $2\boldsymbol{a}$ pointing from the negative to the postive charge is

$$\boldsymbol{E} = \frac{q}{\gamma^2}\left[\frac{\boldsymbol{r}-\boldsymbol{a}}{|\boldsymbol{r}-\boldsymbol{a}|^3}(1+\beta^2 sin^2\alpha_+)^{-3/2} - \frac{\boldsymbol{r}+\boldsymbol{a}}{|\boldsymbol{r}+\boldsymbol{a}|^3}(1+\beta^2 sin^2\alpha_-)^{-3/2}\right], \tag{42}$$

where the origin of the coordinate system (origin of $\boldsymbol{r}$) is halfway between the two charges; $\boldsymbol{r}-\boldsymbol{a}$ and $\boldsymbol{r}+\boldsymbol{a}$ are the field positions with respect to charges $+q$ and $-q$, respectively; and α_+ and α_- are the angles between $\boldsymbol{v}$ and $\boldsymbol{r}+\boldsymbol{a}$ and $\boldsymbol{r}-\boldsymbol{a}$, respectively. All quantities in the above two equations are defined in the moving reference frame.

In finding the dipolar field as the dipole moment $\boldsymbol{p}=2\boldsymbol{a}q$ is formed, the following approximations are used.

$$|\boldsymbol{r}\pm\boldsymbol{a}|^{-n}\approx\frac{1}{r^n}\left(1\mp 2n\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^2}\right). \tag{43}$$

$$\begin{aligned} sin^2\alpha_{\pm} &= 1-\frac{r^2cos^2\alpha\mp 2(ar)cos\alpha cos\theta}{|\boldsymbol{r}\mp\boldsymbol{a}|} \\ &\approx sin^2\alpha\mp\frac{2\boldsymbol{a}\cdot\boldsymbol{r}}{r^2}cos^2\alpha\mp\frac{2(ar)cos\alpha cos\theta}{r^2}. \end{aligned} \tag{44}$$

Here, α is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$ and θ is the angle between $\boldsymbol{v}$ and $2\boldsymbol{a}$. The above equation results from the fact that

$$|\boldsymbol{r}-\boldsymbol{a}|cos\alpha_+ + 2acos\theta = rcos\alpha + acos\theta = |\boldsymbol{r}+\boldsymbol{a}|cos\alpha_-. \tag{45}$$

It should be kept in mind that the vector $2\boldsymbol{a}$ is Lorentz-contracted in the $\boldsymbol{v}$ direction such that $\boldsymbol{a}=\boldsymbol{a}'_{||}/\gamma+\boldsymbol{a}'_{\perp}$, where $\boldsymbol{p}'=2q\boldsymbol{a}'=2q(\boldsymbol{a}'_{||}+\boldsymbol{a}'_{\perp})$ in the (primed) frame stationary with respect to the dipole. With these approximations, Eq. (42) becomes

$$\boldsymbol{E}=\frac{1}{\gamma^2}\left[(1-\beta^2sin^2\alpha)\left(\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}-\frac{\boldsymbol{p}}{r^3}\right)+\frac{2\beta^2(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}cos^2\alpha}{r^5}\right]. \tag{46}$$

Eq. (46) is a dipolar-like equation. Ignoring the second term in square brackets, the first term is the electric field of a dipole with a dipole moment given by $(1-\beta^2sin^2\alpha)\boldsymbol{p}/\gamma^2$. However, the field as a whole is not strictly that of a dipole, and merely replacing $\boldsymbol{p}'$ with $\boldsymbol{p}$ does not give the correct field. However, once again the real problem is the induced field − a magnetic field this time. The magnetic field is given by

$$\boldsymbol{B}=\frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\approx\frac{1}{c^2}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})(\boldsymbol{v}\times\boldsymbol{r})}{r^5}-\frac{\boldsymbol{v}\times\boldsymbol{p}}{r^3}\right], \tag{47}$$

where the slow-motion version is displayed. Following the procedure used for Eq. (39), the magnetic field can be written as

$$\boldsymbol{B}=\frac{1}{c^2}\left[\frac{3(\boldsymbol{m}'\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}-\frac{\boldsymbol{m}'}{r^3}+\frac{3(\boldsymbol{p}\times\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{v})}{r^5}-\frac{\boldsymbol{m}'}{r^3}\right], \tag{48}$$

where $\boldsymbol{m}'=\gamma\boldsymbol{p}\times\boldsymbol{v}$ is the usual form used for the magnetic dipole of the transformation. This equation is quite similar to Eq. (39), consisting of a modified dipole form, but, like that equation, there is no field parallel to the $\boldsymbol{v}$ direction. Hence, both Eqs. (39) and (48) show that the result of transforming the fields of moving dipoles does not result in a transformed dipole field.

VII. THE MAGNETIZATION-POLARIZATION TENSOR

In this section I am going to argue that the magnetization-polarization tensor may not be Lorentz-invariant. Take, for example, something that is often done in the literature: transforming a magnetic dipole from its rest frame to a moving frame using this tensor, resulting in an electric dipole appearing on the magnetic dipole. From what I have presented above, it should be clear that the resulting electric dipole is not equivalent to an electric dipole formed by a charge separation. Furthermore, such a tensor construct does not seem to be consistent with the tensor's definition.

The magnetization-polarization four-tensor is defined indirectly as that tensor which, when differentiated and contracted, yields the effective current density four-vector due to magnetization and time-varying polarization [3]. That is,

$$J^{\mu}_{eff}=\frac{\partial M^{\mu\nu}}{\partial x^{\nu}}=(\nabla\times\boldsymbol{M}+\frac{\partial\boldsymbol{P}}{\partial t},-c\nabla\cdot\boldsymbol{P}).. \tag{49}$$

Here,

$$x^\mu = (x, y, z, ct), \tag{50}$$

and

$$M^{\mu\nu} = \begin{pmatrix} 0 & M_z & -M_y & cP_x \\ -M_z & 0 & M_x & cP_y \\ M_y & -M_x & 0 & cP_z \\ -cP_x & -cP_y & -cP_z & 0 \end{pmatrix}. \tag{51}$$

The polarization and magnetization are defined as, respectively, [22, 23]

$$\boldsymbol{P} = N < \boldsymbol{p}_{mol} >, \tag{52}$$

and

$$\boldsymbol{M} = N < \boldsymbol{m}_{mol} >, \tag{53}$$

where N is the number of molecules in a given region of a substance and $< \boldsymbol{p}_{mol} >$ and $< \boldsymbol{m}_{mol} >$ are the average electric and magnetic dipoles, respectively, per molecule. The definition is made such that the region involved is much smaller than the scale of measurement and much larger than the size of an individual molecule. The terms $\nabla \times \boldsymbol{M}$ and $\partial \boldsymbol{P}/\partial t$ are the effective magnetization and polarization current densities, respectively, and $-\nabla \cdot \boldsymbol{P}$ is the effective charge density due to polarization. Considering these definitions, it is not clear that substituting individual dipoles for $\boldsymbol{M}$ and $\boldsymbol{P}$ is appropriate for this tensor.

However, there are also problems when the tensor is applied to bulk material. For example, consider a block of substance with uniform and isotropic magnetization. It will be represented by a tensor with only magnetization components. When it is Lorentz-transformed to a frame in relative motion it will become a tensor with both magnetization and polarization components. If the tensor has a magnetization component in the z direction and the magnetic material is moving in the positive x direction, the transformed tensor will contain an electric polarization component directed in the negative y direction. At a great distance from this material it would be expected that the electric field would be like that of an electric dipole, but this will not be the same as the transformed electric field that is calculated directly from the moving magnetic field.

Authors have examined the fields of moving dipoles in terms of decompositions of multipole expansions [6, 7] and have considered the question as to which decomposition yields "the" dipole resulting from the transformation of the magnetization-polarization tensor. However, these fields are clearly not identical to dipole fields. In fact, the transformed fields are identical to the fields expected if you apply Maxwell's equations. (This, of course, has to be the case.) In a frame in which the dipoles are moving, the fields at any point in space are, in general, time-dependent. In the case of a moving magnetic dipole, solving Faraday's law for the time-dependent magnetic field yields the transformed electric field. Applying the corrected version of Ampere's law to the time-dependent electric field of the moving electric dipole results in the transformed magnetic field.

A four-matrix appearing in an equation in special relativity is not a four-tensor if the equation does not remain valid when transformed from one inertial frame to another. In the frame of reference of the definition of the tensor, the fields are consistent with the polarization and magnetization. Transform the tensor to another frame and this is no longer the case. I suspect the reason may be due to the relativity of simultaneity. It was shown above that a moving current loop has no charge separation on it, but rather an excess charge appears to be present due to the relativity of simultaneity. This is due to the finite size of the loop: there are spatial and time differences from one point on the loop to another. In the same way the regions used in defining magnetization and polarization are finite. There are spatial and time differences there also. A quantity of magnetization or polarization is not characteristic of a single event (at a certain time and point in space).

Another problem concerns a Gilbertian magnetic dipole. There is no current involved here and the argument of the relativity of simultaneity is not valid. Nevertheless the magnetization-polarization tensor transformation results in an electric dipole on the magnetic dipole just as it does for a current loop, and, once again, the electric field is not that of a dipole with a charge separation. How is it that a presumably fundamental particle (magnetic monopole) can sport a charge separation? (For that matter, how can a moving electron with its magnetic moment sport a charge separation?) This argument obviously carries over to a substance consisting of Gilbertian dipoles.

Yet another problem concerns the combination of current densities to produce an effective current density. The terms $\nabla \times \boldsymbol{M}$ and $\partial \boldsymbol{P}/\partial t$ are separate three-current densities. Their sum is an effective three-current density but not, in general, the space component of the four-current density [24]. Since the effective four-current density is not a four-vector in this case (and not in general), the components of the magnetization-polarization tensor are not proper components of a genuine Lorentz four-tensor. The easiest way to resolve these dicrepancies (Occam's razor) is to concede that the magnetization-polarization tensor is not a legitimate tensor in special relativity.

VIII. DISCUSSION AND CONCLUDING REMARKS

A global Lorentz transformation of the magnetic field of an at-rest, current-carrying wire to a frame of reference moving parallel to the wire results in a cylindrically symmetric radial electric field centered on the wire. This has apparently lead to the belief that the electric field experienced by a charge moving parallel to the wire is due to the difference in the Lorentz contraction of the spacing of charge carriers versus that of ions as seen in the moving frame ("differential Lorentz contraction"). This expectation is supported, by calculation at least, for fields associated with a wire carrying a current between a source and sink of charge [1]. However, it does not give the correct electric field of a wire as part of a moving current loop.

The relativity of simultaneity approach is able to reproduce the appearance of charge proportional to the current on a moving wire without the presence of a source or sink of charge. The result is there is no actual charge on the wire but only the appearance of charge due to timing differences between the stationary and moving reference frames. An observer located along the wire and moving with respect to it observes a different amount of charge having passed her position compared with other observers located at different points along the wire at the same time in the moving reference frame. To the moving observers there appears to be an excess of charge (positive or negative depending on the motion) on the wire due to timing differences between the moving and stationary frames.

The effect of Lorentz contraction can also be explained in terms of the relativity of simultaneity. An observer moving along the wire in the opposite direction of the current will see more charge carriers having passed her position than a trailing observer at the same time in their reference frame. Hence they will consider the charge carriers to be closer together than the ions, which are not moving as quickly in their frame. The observers must put their observations together to confirm length contraction, so the picture of the observers in the moving frame seeing themselves all equally spaced like birds on a Lorentz-contracted powerline is not correct. (See the example in [20].) In the frame at rest with the wire, the relatively moving observers arrive at the events identified in Fig. 1 at the same time, but in their frame this arrival is not simultaneous, resulting in the appearance of extra charge on the wire.

There is a problem with the idea that a charge separation on a moving Amperian magnetic dipole produces an electric dipole on the magnetic dipole due to differential Lorentz contraction and that a magnetic dipole appears on a moving electric dipole. If so, then why are the resulting fields due to the Lorentz transformation not dipolar? If there were a charge separation on a moving current loop, the electric field would look like that of an actual dipole with loops of field lines running from the positive side of the loop to the negative side, but that is not the case. Rather, the electric field is the same as that found by applying Faraday's law to the time-varying magnetic field due to the motion of the current loop.

A charge separation due to differential Lorentz contraction cannot explain the form of the electric field due to a moving current loop; it should result in a dipolar-like field – a result contradicted by transforming the fields directly. A similar problem arises for a moving electric dipole with the additional difficulty that there is no magnetic current on an electric dipole to which a differential Lorentz contraction can be applied. Rather, it seems clear to me that the practice of treating the dipole moments obtained by applying the Lorentz transformation to the magnetization-polarization tensor like actual dipoles in subsequent calculations is in error. This problem extends to bulk magnetic and electrical materials.

The problem may arise from the definition of this tensor. Each component of the tensor covers an average of a region of space such that events are separated in space and time over the region. The components are therefore not representative of the electrical or magnetic properties of a single event in spacetime and the relativity of simultaneity between different events in the region may become important. Additionally, the sum of the magnetization and polarization contributions to the current density is not, in general, a valid spatial component of the current density four-vector, and a tensor based on this invalid four-vector is also invalid. I propose that the magnetization-polarization tensor is not a valid Lorentz four-tensor.

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