The Trouton-Noble paradox and the von Laue current

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I review the resolution of the Trouton-Nobel by Max von Laue and present an approach using the tensor formulation of Special Relativity on an equivalent but slightly different model than the original right-angle lever treated by von Laue. The results of this approach can be interpreted to be compatible with the ideas of von Laue involving momentum flow.

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I. INTRODUCTION

Issues concerning linear and angular momentum in Special Relativity have a long history in physics [1–9]. In particular, much has been published about the Trouton-Noble Paradox involving forces acting on a right-hand lever (or ”bracket” as some refer to it) [2–4, 10–13]. In the lever's rest frame external forces are applied such that the lever is in equilibrium for both translation and rotation. However, when the lever is moving, the relativistic view from its original rest frame makes it appear to be acted on by a net torque, which according to Newtonian physics should produce a rotational acceleration of the lever. The fact that the lever does not rotate is the gist of the paradox.

The paradox was early on addressed by Max von Laue, who used the idea of a flow of momentum due to stresses in an extended body to resolve the paradox [3]. (The idea of momentum flow in relativity was introduced by Planck [1].) According to Janssen [14], who did an in-depth study of the history of the paradox, there has been no real improvement over Max von Laue's treatment.

My purpose here is to review von Laue's [3] resolution of the paradox and to relate it to a similar result using the tensor approach to relativity. In this approach I replace the right-hand lever with a square stressed by external forces in the same manner as the lever. In the moving frame it turns out there are two contributions to the angular momentum. There is a constant angular momentum due the motion of the center of mass of the square with respect to a fixed point. The second contribution is time dependent and opposite to that found by von Laue. It arises from the stresses in the square and, in my opinion, is consistent with the argument of von Laue.

II. THE NEWTONIAN VIEW OF THE FORCE-LEVER MODEL

Fig. 1 (left side) shows the L-shaped right-angle lever in its rest frame, which I will call S. The sides of the L are both length l, and forces of magnitude F are applied to points A and B at the tips of the L, with the force at B in the positive x direction and the force at A in the positive y direction. Equal and opposite forces to these are applied to the bend of the L, originally at the origin of the rest frame at point O, to maintain both translational and rotational equilibrium. In the rest frame of the L, the vector sum of the forces is zero and the torque about any point (either on the lever or off in space) is zero. (For example, the torque about point O in the figure would be $Fl - Fl = 0$.)

Let the lever move to the right in the figure (positive x direction) with speed v. The now moving lever is at rest in a new reference frame, called S', moving along with it. In Newtonian theory the relationship between these two reference frames is determined by the Galilean transformation, in which is assumed absolute space and time. Hence the lever still has its same dimensions in the S' frame as viewed from the S frame, and the forces are acting at the same time in the S' frame as in the S frame. That is, if the forces were temporary and acted at the same time in the S frame, the same situation would be seen in the S' frame. There would be no acceleration of the lever as the forces at all times add to zero. There would be no rotational acceleration as the torques at all times add to zero. No paradox is seen from the point of view of Newtonian physics. (I have heard it erroneously stated that the paradox was resolved once relativity theory was formulated. In fact it arose as a result of the theory.)

III. THE RELATIVISTIC VIEW OF THE FORCE-LEVER MODEL

In the relativistic view a paradox arises. The Lorentz transformation, based on the constancy of the speed of light in all inertial reference frames (IRF – frames in which the first law of Newton holds), results in both space and time being altered from an observer's perspective, depending on her motion. Say the forces in Fig. 1 are constant. When viewing the lever in the “moving” S' frame from the “stationary” S frame (“moving” and “stationary” being relative terms), the distance between points O and A appears to be shorter due to length contraction. (It should be kept in mind that this is not what a single observer in the S frame would actually see. The contraction would have to be verified by at least two observers strategically placed in frame S. See Rindler [15] for an account.)

The situation for constant forces acting on the lever in the relativistic view for the moving frame is also shown in Fig. 1. Not only is the length from O to A shortened from l to l/γ, where γ is the Lorentz factor in Special Relativity given by

$$\gamma = \sqrt{\frac{1}{1 - \dfrac{v^2}{c^2}}}, \tag{1}$$

but the Lorentz transformation also reduces the force acting at A by the same factor such that it now equals F/γ.

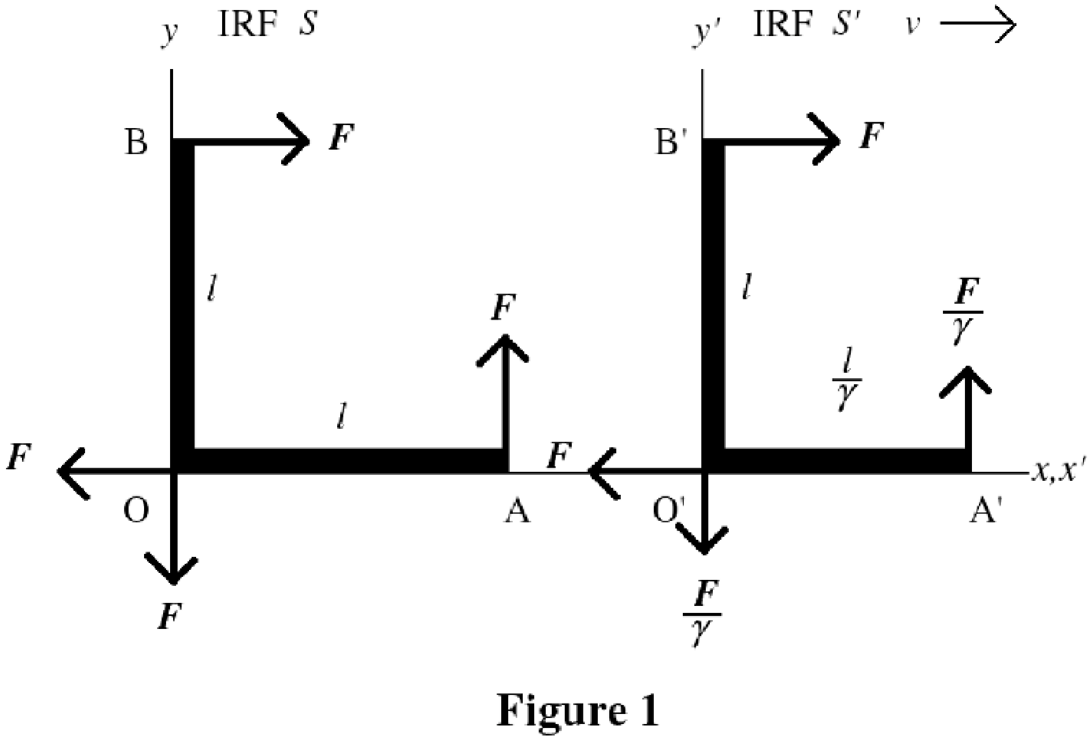

FIG. 1. Assumed Right-Angle Lever Forces for Rest and Moving Frames.

This produces a net torque on the lever of

$$\boldsymbol{\tau}_{tn} = l\hat{\boldsymbol{\jmath}} \times F\hat{\boldsymbol{\imath}} + \frac{l}{\gamma}\hat{\boldsymbol{\imath}} \times \frac{F}{\gamma}\hat{\boldsymbol{\jmath}} = -Fl\frac{v^2}{c^2}\hat{\boldsymbol{k}}, \tag{2}$$

which is the troublesome torque found by Trouton and Noble. Yet, the lever certainly doesn't rotate in the frame S'. The angular momentum of the lever does not change as time goes on, increasing in the negative z direction as the mathematics above would indicate. If the torque is not producing an angular acceleration of the lever, then what is its effect?

IV. THE APPROACH OF VON LAUE

Planck proposed that an energy flow also resulted in a flow of momentum [1]. If ρ_E is the energy density flowing with velocity v, then the momentum flow according to Planck would be

$$\frac{d\boldsymbol{g}}{dt} = \frac{\rho_E \boldsymbol{v}}{c^2}, \tag{3}$$

where $\boldsymbol{g}$ is the momentum density. The force acting at point B is doing work, since point B is moving in the same direction in that frame as the force. The power being supplied at point B′ as seen in S is Fv, so there is an input of energy to the lever at that point. According to von Laue this means there is an energy current from point B′ to O′ given by $Fv/(\Delta x \Delta z)$, where the cross-sectional area of the arm from O′ to B′ is $\Delta x' \Delta y'$. Integrate this over the volume to get the energy flux as follows,

$$\int_{V'} \frac{Fv}{\Delta x' \Delta z'} dV' = \frac{Fv}{\Delta x' \Delta z'} \Delta x' \Delta z' l = Fvl. \tag{4}$$

The momentum flux will therefore be, according to Planck,

$$\frac{dp'}{dt'} = \frac{Fvl}{c^2}, \tag{5}$$

where p' is the momentum.

The formula for the angular momentum L due to the linear momentum of a body with respect to a point is $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, where $\boldsymbol{r}$ is the position vector from the point to the center of mass of the body and $\boldsymbol{p}$ is the body's linear momentum. Similarly, the momentum flux in Eq. (5) will produce an angular momentum about a chosen point. Von Laue chooses a point along the x axis to the left of the moving lever, say point O, the original position of the lever's "elbow". Therefore the vector $\boldsymbol{r}'$ in this case will be the position vector from O to O′ (Fig. 2) plus the vector $vdt'\hat{\boldsymbol{\imath}}$; in other words, $\boldsymbol{r}'$ is increasing in time, meaning $\boldsymbol{L}'$ is also increasing in time. This argument leads to the equation for the increase in $\boldsymbol{L}'$ over the time dt'.

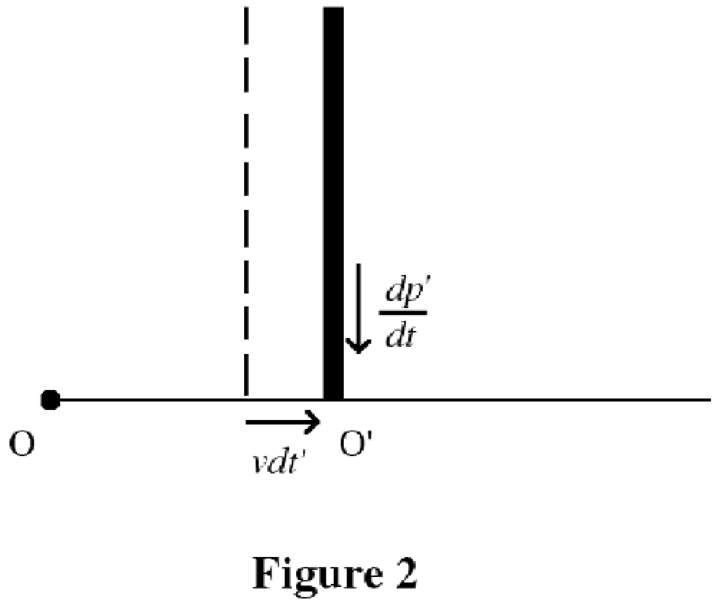

FIG. 2. Von Laue's Argument for How Angular Momentum Changes in the Right-Hand Lever.

$$d\boldsymbol{L}' = vdt'\hat{\boldsymbol{\imath}} \times \frac{dp'}{dt}(-\hat{\boldsymbol{\jmath}}) = vdt'\hat{\boldsymbol{\imath}} \times \frac{Fvl}{c^2}(-\hat{\boldsymbol{\jmath}}) = -\frac{Flv^2}{c^2}dt'\hat{\boldsymbol{k}}. \tag{6}$$

The time rate of change of the angular momentum is the torque, which according to the above equation would be

$$\boldsymbol{\tau}'_{vL} = \frac{d\boldsymbol{L}'}{dt'} = -\frac{Flv^2}{c^2}\hat{\boldsymbol{k}}. \tag{7}$$

This is the same as the torque apparently seen in S' due to the application of external forces to the lever. Von Laue argues that the applied torque seen in the S' frame results in a current of energy and momentum, not in the rotation of the lever. He then imagines the lever to be held in some sort of external case (or "shell" as he puts it) responsible for applying the forces to the lever. The case will experience equal and opposite forces, hence the energy and momentum currents will be equal and opposite to those in the lever. Integrating these currents over the lever and the case will result in zero, he argues, just as integrating an electrical current around a circuit gives zero.

V. APPLICATION OF TENSOR FORMALISM TO THE PARADOX

Instead of the original right-angle lever, I'm going to consider a rigid square of length l on a side in the x-y plane with the bottom of the square resting on the x axis. The thickness of the square is dz. Forces of magnitude F are applied parallel to the sides to produce stress in the square. The top of the square has a force F acting tangent to the x-z plane in the positive x direction. There is an equal and opposite appled force to the bottom of the square. The right side of the square has a force F parallel to the side acting in the positive y direction with an equal and opposite force on the left side of the square (Fig 3). These forces create stress in the square of magnitude $\sigma_{xy} = \sigma_{yx} = F/Ldz$. The stress-energy tensor should then be (taking the subscript 4 to indicate the time component)

$$(T_{\mu,\nu}) = \begin{pmatrix} 0 & \sigma_{xy} & 0 & 0 \\ \sigma_{yx} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho c^2 \end{pmatrix}, \tag{8}$$

where σ_{xy} is the stress on a plane perpendicular to the y axis from a force in the x direction, σ_{yx} is the stress on

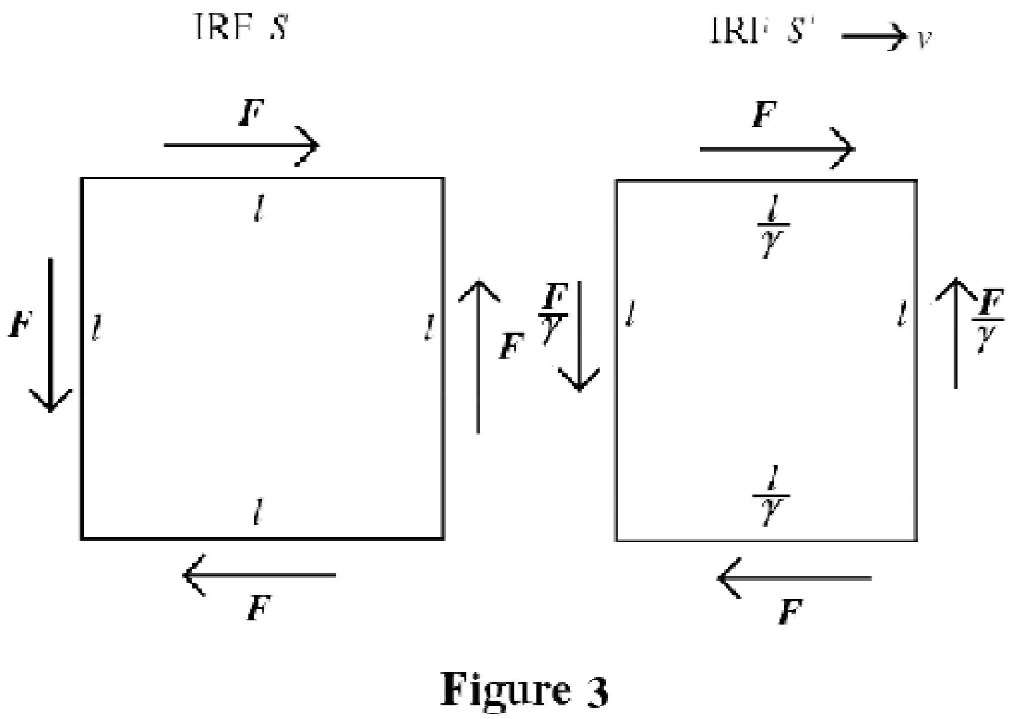

FIG. 3. The Stressed Square Model

a plane perpendicular to the x axis from a force in the y direction, and ρc^2 is the relativistic energy density. Let the square move in the positive x direction. To view the situation of the moving square from the original (S) frame, you must perform a Lorentz transformation. If Λ is the Lorentz transformation expressed as a four-tensor, then the transformed stress-energy tensor is found as follows.

$$T' = \Lambda T \Lambda^T, \tag{9}$$

where Λ^T is the transposed form of Λ (which is the same since Λ is a symmetric tensor). Λ has the form

$$(\Lambda_{\mu,\nu}) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\frac{v}{c} & 0 & 0 & \gamma \end{pmatrix}, \tag{10}$$

The Lorentz-transformed tensor is therefore

$$(T_{\mu',\nu'}) = \begin{pmatrix} \gamma^2\rho v^2 & \gamma\sigma_{xy} & 0 & \gamma^2\rho vc \\ \gamma\sigma_{yx} & 0 & 0 & \gamma\frac{v}{c}\sigma_{yx} \\ 0 & 0 & 0 & 0 \\ \gamma^2\rho vc & \gamma\frac{v}{c}\sigma_{xy} & 0 & \gamma^2 c^2\rho \end{pmatrix}. \tag{11}$$

The elements of this tensor can be interpreted as follows. $T_{1'1'} = \gamma^2\rho v^2$ is the dynamic pressure (the usual Bernoulli pressure ρv^2 times γ^2). $T_{4'4'} = \gamma^2\rho c^2$ is the energy density. These are both multiplied by the square of γ: One γ is due to Lorentz contraction increasing the density, and the other is due to the relativistic increase in mass (or, equivalently, in momentum or in energy). Note the mechanical stresses are increased by γ. Finally, $T_{1'4'}/c = \gamma^2 v\rho$ is the momentum density in the x' direction, and $T_{2'4'}/c = \gamma\frac{v}{c^2}\sigma_{yx}$ is the momentum density in the y' direction.

The angular momentum about the elbow of the lever in S' (which will be an axial vector parallel to z') can be found from the transformed stress-energy tensor by integrating over the volume of the square as follows.

$$L'_z = \frac{1}{c}\int_{V'}(x'T_{2'4'} - y'T_{1'4'})dx'dy'dz' = \frac{1}{c}\int_{V'}\left[x'(\gamma\frac{v}{c}\sigma_{yx}) - y'(\gamma^2\rho vc)\right]dx'dy'dz' \tag{12}$$

I will evaluate the second integral in the above equation first. Note that $\rho dx'dy'dz' = M/(l^2dz')dx'dy'dz' = M/l^2dx'dy'$. The equation becomes

$$-\frac{1}{2}\frac{\gamma^2 Mv}{l^2}\int_{V'}y'dx'dy' = -\frac{1}{2}\gamma Mvl. \tag{13}$$

This is nothing more than the angular momentum about point O$'$ due to the linear momentum of the square (where the lever arm is the distance between point O$'$ and the center of mass of the square parallel to the y' axis. Note that it is in the $-z$ direction, as is necessary. This angular momentum is constant in time, meaning no torque is associated with it.

Now to evaluate the first integral in the equation for L'_z. Noting that $\sigma_{yx} = F/(ldz')$, the integral is, taking the integral over x' as an indefinite integral for now,

$$\frac{1}{c}\gamma\frac{v}{c}\frac{F}{l}\int_{V'}x'dx'dy' = \frac{1}{2}\gamma\frac{v}{c^2}Fx'^2. \tag{14}$$

The position x' is a function of time. Using the Lorentz transformation and the chain rule you find that $dx'/dt' = (dx'/dx)(dt/dt')(dx/dt) = \gamma/\gamma(dx/dt) = v$. Therefore, taking the time derivative of the above equation to get the torque, you have

$$\boldsymbol{\tau'_z} = \gamma\frac{v}{c^2}Fvx' \to \frac{v^2}{c^2}Fl, \tag{15}$$

where at the last I have identified position x' with l/γ.

This is the opposite of the result of von Laue, and it would be tempting to assume this torque cancels the applied torque, but that would be incorrect. The square is stressed in opposition to the applied forces so that this torque is due to internal forces. However, this result is compatible with von Laue's idea of a momentum flow using the following argument. The stresses result in the reaction forces acting on whatever agent (such as a case) is responsible for the applied forces, producing a counter torque. Von Laue's interpretation would probably be that this reaction torque results in an energy and momentum flow into the case, equal and opposite to that from the case into the square.

It appears from this development that the torque perceived in S when examining S' goes into the time rate of change of the angular momentum associated with the momentum density, which is a flow of momentum between between the top and bottom of the square. Momentum and energy flow into the square at the top and out of the square at the bottom. This is little different, physically, from an energy-momentum flow into the lever at B′ and out of the lever at O′ as maintained by von Laue. At the same time an equal and opposite momentum density flows into the case holding the square.

Von Laue introduced what he called a "shell" in which the lever is mounted. The shell is responsible for the forces on the lever. In the case of the square you might have a slightly rhombus-shaped mounting of side l that could be stressed into a square frame and fitted onto the square in such a way as to provide the necessary stress. Treating this mounting like the square, you would merely change the sign of the stresses in the stress-energy tensor. When the steps above are repeated for this mounting, you again get the constant angular momentum in the $-z$ direction, but also a time-dependent angular momentum that results in a torque in the *positive* z direction equal in magnitude to that on the square. The angular momentum of the entire contraption therefore remains constant.

VI. DISCUSSION AND CONCLUSION

Momentum and energy flow when nothing appears to be moving is not an easy concept to grasp. However, imagine a worker pushing a crate across a factory floor. He is pushing near the top of the crate and the crate is sliding over the floor against friction. Of course, the worker is doing work on the crate as a result of the power he is supplying. Where is this energy input going? Not to the crate if it is moving at a constant speed. The Newtonian way of looking at this is that the worker is doing positive work on the crate, and the floor through friction is doing an equal and opposite amount of work on the crate such that the total work on the crate equals zero. To say energy is flowing through the stressed crate to the floor seems a bit far-fetched.

But bring in the relativistic relationship between matter and energy – our old friend $E = mc^2$. The floor and the bottom of the crate are heating up slightly due to the sliding of the crate, and heat is a form of energy. Hence the floor and the crate are gaining mass! Meanwhile the worker is expending energy and losing mass. When looked at in this fashion, it certainly appears there is an energy (mass) flow from the worker to the floor.

Now consider a worker giving the crate a single shove, moving it just a short distance. This will stress the crate, but the stress is not instantly transmitted from the top of the crate to the bottom. A pulse of stress is transmitted down the crate – a type of mechanical wave. It is clear how the energy is transmitted in this case. The wave pulse carries energy downward toward the floor. When it reaches the floor, the crate slides and heat is generated. It is also clear that a net torque is acting on the crate as a whole between the time of the shove and the reaction at floor level (where there are both frictional and normal force impulses).

The case of the right-hand lever and stressed square are, of course, different from the pushed crate in the sense there is no differential motion between objects. However, there is the relativity of simultaneity. Replace the constant forces acting on the right-hand lever with impulses that last a very short time, as was done by Shapiro [11]. As in Fig. 1 the x' and y' axes lie along the arms of the lever. Arrange the motion of the lever such that x and x' and y and y' axes coincide at $t = t' = 0$. When this happens apply equal impulses at the same time ($t = 0$) in the S frame at points A and B. These are two events in spacetime. In four-vector notation, the one for point B occurs at $(0, l, 0, 0)$ and the one at point A occurs at $(l, 0, 0, 0)$ in the frame S. In the S' frame these events occur at $(0, l, 0, 0)$ and $(\gamma l, 0, 0, -\gamma(v/c)l)$, respectively. Hence the event at A occurs at a time $-\gamma(v/c^2)l$ earlier in the S' frame than it does in the S frame and there is a net torque acting on the lever (and again when the force at B occurs later). But the lever does not rotate because the momentum/energy flow accounts for the changing angular momentum.

Finally, what about the question of so-called "hidden momentum" in the lever (or square) [12]? The momentum flow in this case is very different from the claim of hidden momentum in, for example, the charge-magnet model of Shockley and James [5]. In that model there is no external force identified that is related to any sort of momentum flow, unlike the applied forces in the Trouton-Noble paradox. I have shown that there is no hidden momentum in the Shockley-James model or in some other models for which that is claimed [17, 18]. The problem has to do with the misapplication of the center of energy theorem. The center of energy is taken as stationary in these models when, in fact, it is not. I conclude there is no hidden momentum in the Trouton-Noble paradox of the kind claimed for certain electromagnetic systems.

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