A look at some paradoxes in electromagnetic theory

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Abstract

In this paper I attempt to resolve several paradoxes in electromagnetic theory that continue to be the subjects of dispute among some physicists, although the conventional wisdom of the physics community appears to consider them solved. The concept of hidden momentum has been employed to explain a number of these puzzles, but the existence of hidden momentum has been seriously questioned. I show several paradoxes can be resolved without appealing to hidden momentum.

I. INTRODUCTION

In 1891 J.J. Thompson pointed out an apparent paradox[1] where electromagnetic systems at rest could contain non-zero electromagnetic momentum. In 1967, Shockley and James[2] examined a charge-magnet system and were puzzled by the apparent lack of momentum conservation. There was linear momentum in the electromagnetic fields but the system was at rest, containing no mechanical momentum. They thought there had to be a hidden form of mechanical momentum in the magnet to preserve momentum conservation. The idea of this so-called "hidden momentum" has been employed to solve numerous problems and paradoxes in addition to the original paradox Shockley and James identified[2].

Hidden momentum can also be a problem in solving paradoxes. Vaidman[3] solved a so-called paradox of a current loop moving in a uniform electric field. Even if the resolution were correct in itself, the presence of hidden momentum in the loop claimed by Vaidman, spoils the the resolution.

The Mansuripur paradox[4] involving a charge-current loop system provoked an extended discussion in the literature and was thought to be solved by invoking hidden momentum in the current loop[5]. Mansuripur pointed out there was no torque observed on an Amperian magnet in the vicinity of an electric charge in the rest frame of the charge-magnet system, but there was a torque seen by an observer moving with respect to the system. He argued this situation negated the Lorentz force law, which he said should be replaced by the force law of Einstein and Laub[6]. His critics[5] argued that if you take hidden momentum in the magnet into account you can preserve the Lorentz force.

Aharonov and Casher[7] proposed that there was an interaction between Amperian dipoles (such as a current loop) and a line of electric charge analogous to the Aharanov-Bohm effect[8]. In the latter effect there is a phase difference between the wave functions of charged particles passing opposite sides of a solenoid. In the Aharonov-Casher effect the phase difference is between the wave functions of neutral Amperian dipoles passing opposite sides of a line of charge. It was proposed this effect could be seen experimentally for neutrons, assuming they behaved as Amperian dipoles. In both effects it is supposed there is no force between the solenoid and the charges on the one hand and none between the line of charge and the dipoles on the other.

Although the Aharonov-Bohm effect has been experimentally demonstrated (see, for example, Chambers[9]), Boyer[10] has disputed the quantum nature of the Aharonov-Casher effect by calculating a force on a dipole passing a line of charge, assuming an electric dipole is induced on the moving magnetic dipole. In response, Aharonov *et al.*[11] claimed hidden momentum in a magnetic dipole immersed in an electric field[7] was responsible for neutralizing the force on the induced electric dipole.

A few authors have argued that hidden momentum in magnets subject to an electric field does not exist[4,12–15] This calls for an effort to revisit the various paradoxes supposedly solved by hidden momentum. In this paper I address the paradoxes described above and show there is a resolution for each one without appealing to hidden momentum.

II. THE PARADOX TREATED BY VAIDMAN

In this paradox (which I will call the "Vaidman paradox" for brevity, although it was treated prior to the paper by Vaidman by Bedford and Krumm[16] and by Namias[17]) there is a current loop oriented parallel to the y-z plane with its magnetic moment $\boldsymbol{\mu} = \mu\hat{\boldsymbol{i}}$ pointed in the positive x direction. The space containing this magnet is filled with a uniform electric field in the positive z direction ($\boldsymbol{E} = E\hat{\boldsymbol{k}}$). This ring of current is moving with a constant velocity in the positive x direction ($\boldsymbol{v} = v\hat{\boldsymbol{i}}$) in the lab frame S. In the moving S' frame – the rest frame containing the current loop – the transformed fields are

$$\boldsymbol{E}' = \gamma\boldsymbol{E} = \gamma E\hat{\boldsymbol{k}}, \tag{1}$$

and

$$\boldsymbol{B}' = -\gamma\boldsymbol{v} \times \boldsymbol{E}/c^2 = \gamma(v/c^2)E\hat{\boldsymbol{j}} \tag{2}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2}. \tag{3}$$

The presence of the magnetic field in the S' frame means there is a torque on the current loop given by

$$\boldsymbol{\tau}' = \boldsymbol{\mu}' \times \boldsymbol{B}' = \mu(v/c^2)E\hat{\boldsymbol{k}}, \tag{4}$$

where the term on the far right uses the slow-motion approximation; that is, $\gamma = 1$ ($v << c$), meaning $\boldsymbol{\mu}' = \boldsymbol{\mu}$ and $\boldsymbol{E}' = \boldsymbol{E}$. This approximation will be employed for the rest of this section.

The problem is there is no magnetic field in the S frame and therefore (presumably) no torque. Why is it that an observer in S' records a torque that is not observed in the S frame?

Vaidman resolved the paradox for his model (iii), which consists of a "charged incompressible liquid moving inside a neutral tube". Franklin[13] pointed out a flaw in that resolution. There should be no electric field inside the charged liquid and, with no electric field, there would not be a magnetic field from a Lorentz transformation of the electric field either. There will be induced charges on the conducting material due to its immersion in the electric field, but these are stationary in S' and so do not interact with the magnetic field. For this magnet model there is no torque in either the S or S' frames. However, Vaidman considers there to be linear hidden momentum in the current loop. I will show how this spoils his resolution even if it were valid. There is a general resolution, described below, that applies to any Amperian dipole model, including a current loop like that of Shockley and James[2] where there is no conducting material.

The torque on the ring is the time rate of change of its angular momentum in the rest frame (S') of the loop. The hidden linear momentum, if present, appears in the angular momentum four-tensor in the y-ct slot with its negative in the ct-y slot. (See the following equations.) These slots contain the y component of the linear momentum multiplied by ct, where t is the time coordinate in the frame of the tensor. The time integral of the magnetically-produced torque and its negative appear in the x-y and y-x slots, respectively, home to the z component of the angular momentum. When the angular momentum four-tensor is Lorentz-transformed to the S frame, you obtain a four-tensor containing the transformed z-component of the S'-frame torque plus a contribution from the hidden linear momentum. If the z component of the angular momentum is zero in the S' frame, it will not be zero in the S frame and the resolution of the paradox breaks down. The details are as follows.

The angular momentum four-tensor is given by

$$L^{\mu\nu} = \begin{pmatrix} 0 & L_z & -L_y & mcx - ctp_x \\ -L_z & 0 & L_x & mcy - ctp_y \\ L_y & -L_x & 0 & mcz - ctp_z \\ mcx + ctp_x & mcy + ctp_y & mcz + ctp_z & 0 \end{pmatrix}. \tag{5}$$

Here, $\boldsymbol{L}$ is the angular momentum of the system, $\boldsymbol{p}$ its linear momentum, m is the system mass, (x, y, z) is the point about which the angular momentum is taken, c is the speed of light, and t is the time in the rest frame of the system. The hidden linear momentum in the ring in the S' frame is given by Vaidman as[3]

$$\boldsymbol{P}_{hidden} = -\frac{1}{c}\int \phi \boldsymbol{J} dV = \boldsymbol{\mu} \times \boldsymbol{E}/c^2 = -\mu E/c^2 \hat{\boldsymbol{j}}), \tag{6}$$

at $t = 0$ ($t = t'$ in the slow-motion approximation and the S and S' axes are taken to coincide at $t = 0$) when the (possibly rotating) magnetic moment is parallel to the x axis. $\boldsymbol{J}$ is the current density in the loop and $\phi = -zE$ is the electric potential. (According to the above equation, the hidden momentum will change direction if $\boldsymbol{\mu}$ rotates.) According to Vaidman the angular momentum about the center of the loop at $t = 0$ is zero. The angular momentum four-tensor in S' at $t = 0$ is therefore

$$dL^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu E dt/c \\ 0 & 0 & 0 & 0 \\ 0 & -\mu E dt/c & 0 & 0 \\ . & & & \end{pmatrix} \tag{7}$$

The Lorentz-transformed four-tensor will contain

$$dL_z = -(v/c^2)\mu E dt \tag{8}$$

in the x-y slot. The time derivative of this equation gives

$$dL_z/dt = -(v/c^2)\mu E, \tag{9}$$

such that the torque is nonzero in the S frame due to the hidden momentum. So, we are back to the paradox of there being a torque in one frame and not in another.

The problem is no hidden momentum exists in the ring in S'. I have shown[14] that you cannot apply an electric field to a magnet without imparting mechanical linear momentum to it unless the magnet is held stationary by an external agent. In that case the external agent is the recipient of the linear momentum. In both cases electromagnetic linear momentum is produced which is equal and opposite to the mechanical linear momentum. If you let the magnet gain linear momentum, you will need to move to its new rest frame to see the mechanical momentum is zero and the electromagnetic momentum is not. This is what Shockley and James[2] identified as a paradox, but it was really just viewing the system in a rest frame different from that in which the electric field was applied to the magnet.

To get the correct resolution of Vaidman's paradox, consider the following scenario. Imagine there is an observer in a uniform magnetic field. The magnetic field is into the plane of the page and there is a nonconducting rod with two equal and opposite charges at the ends moving from left to right with the positive charge leading. An observer moving with the rod sees an electric field due to the Lorentz transformation of the magnetic field. She will see the electric field directed upwards on the page such that there is an upward force on the positive charge and a downward force on the negative charge. There will be a torque acting on the rod with its axis directed out of the plane of the page. An observer at rest with the magnetic field, however, does not detect an electric field. Why then should he see a torque acting on the rod? But he does.

No one to my knowledge considers the above scenario to be a paradox – just the manifistation of the Lorentz force. But the Lorentz force is due to a Lorentz transformation exactly like the case where the current loop is moving through a uniform electric field and experiencing a torque. In other words, this so-called paradox is not a paradox at all.

For the dipole model of Shockley and James[2] there will be a torque acting on the dipole in the S' frame. There will also be a torque seen to be acting on the dipole in the S frame. In the fully relativistic case the torque in the S frame is multiplied by γ. However, the mass and therefore the moment of inertia also increase by a factor of γ so the angular motion is the same in both frames.

This argument brings up another point. It should be clear that you cannot create an interaction in a system where there is none by merely performing a Lorentz transformation. To do so would violate the principle of relativity. Neither can you Lorentz-transform away an interaction in a system. The interaction involving the magnetic field and the magnet in the S' frame cannot be transformed away by observing the system in the S frame.

III. THE PARADOX POSED BY MANSURIPUR

The paradox posed by Mansuripur in 2012[4] generated a sensation among a lot of physicists who worked in electromagnetic theory[5]. A flurry of comments and papers resulted, most of which appealed to hidden momentum as the solution to the paradox[18–21]. But if hidden momentum is not present in a charge-magnet system, some other resolution must be found.

In this paradox you have a charge in the vicinity of a magnet. In the rest frame of the charge-magnet system, there is only an electric field at the location of the magnet. If the current loop is neutral, there will be no force or torque on the magnet. However, imagine the system is moving in the lab frame where there is an observer. The observer, according to the usual view, detects an electric field due to a charge separation – that is, a dipole – on the magnet. The observer also sees the (transformed) electric field of the charge. Therefore, there should be a torque acting on the electric dipole and thus on the magnet in his frame that is not seen in the rest frame of the charge-magnet system.

Mansuripur claimed that the Lorentz force should be replaced with that of Einstein-Laub[6] to solve the paradox. But, if there is no charge separation on the magnet, there will be no torque observed in either frame and no need to invoke the Einstein-Laub force. The resolution involves the recognition that there is no electric dipole induced on a moving magnetic dipole[22] as was recently also pointed out by Franklin[23].

Imagine a rectangular current loop in frame S' with its length l' parallel to the x axis and moving in the negative x direction with speed v in the lab frame S. (See Figure 1.) A positive current I' is circulating in the rest frame of the loop such that it is flowing to the left in the upper side of the loop. There are n' positive charge carriers per unit length in the loop, each with a charge e, and an equal linear density of negative ($-e$) fixed ions. The average drift speed of the charge carriers is u' in the loop's rest frame, such that an observer moving along with the loop sees a positive current given by

$$I' = \gamma_{u'} e n' u', \tag{10}$$

with

$$\gamma'_u = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}}. \tag{11}$$

In the lab frame the drift speed in the upper wire due to relativistic velocity addition would be

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} - v = \frac{(c^2 - v^2)u'}{c^2 + u'v}, \tag{12}$$

and a current I exists given by

$$I = enu = e(n'\gamma_{u+v})\frac{(c^2 - v^2)u'}{c^2 + u'v} = \frac{\gamma_{u'}}{\gamma_v} e n' u', \tag{13}$$

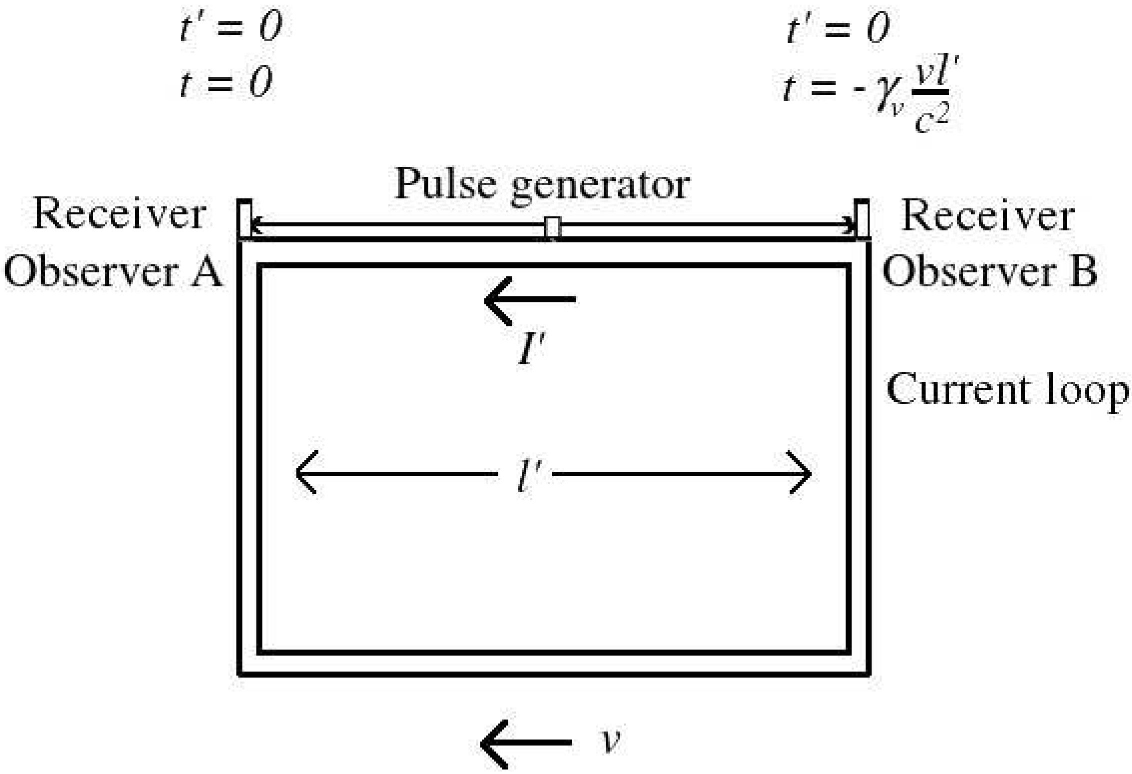

FIG. 1. The apparatus described in the text as observed in its rest frame (S'). It is moving in the negative x direction with speed v in the S frame. (The primed values refer to the moving frame.)

where

$$n = n'\gamma_{u+v} = n'\frac{1}{\sqrt{1-\frac{(u+v)^2}{c^2}}} = n'\frac{c^2+u'v}{\sqrt{(c^2-v^2)(c^2-u'^2)}} = n'\gamma_{u'}\gamma_v\frac{c^2+u'v}{c^2}, \tag{14}$$

and

$$\gamma_{u+v} = \frac{1}{\sqrt{1-\frac{(u+v)^2}{c^2}}} = \frac{c^2+u'v}{\sqrt{(c^2-v^2)(c^2-u'^2)}}, \tag{15}$$

Now imagine that there are observers in the lab frame, lined up to observe the current in the upper wire as it passes each of their positions. Borrowing an example of the relativity of simultaneity from Wolfgang Rindler's book, *Relativity: Special, General, and Cosmological*[24], imagine that there is a laser pulse generator located halfway along the wire which projects identical pulses of sufficient briefness at the same time in opposite directions toward receivers on each end of the wire. In the lab frame the pulse will approach the leading receiver at a relative speed of $c - v$ and the trailing receiver at a relative speed of $c + v$. The distance the pulses travel will be the Lorentz-contracted distance $l'/2\gamma_v$, where

$$\gamma_v = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{16}$$

Hence, observer B who is abreast of the trailing receiver when the pulse arrives will record a time of arrival of

$$t = \frac{l'}{2\gamma_v(c - v)} - \frac{l'}{2\gamma_v(c + v)} = \gamma_v \frac{vl'}{c^2} \tag{17}$$

earlier than that of observer A at the leading receiver when the pulse arrives there. Therefore, by her estimation, an amount of charge given by

$$q = I(t - 0) = \left(\frac{\gamma_u'}{\gamma_v} en'u'\right) \left(\gamma_v \frac{vl'}{c^2}\right) = \gamma_u \frac{uv}{c^2} enl \tag{18}$$

has flowed past her position before the observer at the leading receiver sees the arrival of the pulse, using Eq. (13) for I. That is, observer A has not seen this charge pass his position when the pulse arrives.

The observers in the lab frame must conclude, upon comparing notes, that there was an excess charge on the wire when the pulses arrived at their positions compared to the situation in the S' frame where the pulses arrived simultaneously. They find the excess linear charge density on the wire to be given by

$$e\Delta n = \frac{q}{l'/\gamma_v} = \gamma_v \gamma_u' \frac{u'v}{c^2} en' \tag{19}$$

in their reference frame. They realize that this extra charge was not present in the traveling frame and *was only seen in their frame due to the effect of the relativity of simultaneity.* Hence this is not a real buildup of charge on one side of the loop. The appearance of extra charge is due to the fact the arrival of the pulses was not simultaneous in both reference frames. This extra charge density is what is needed to give the improperly deduced radial electric field found by a misapplication of Gauss' law to a moving line of current (by applying the law as if both ends of the cylindrical Gaussian surface have the same time in both the moving and at-rest frames[22]).

The same conclusion was reached by Franklin[23] by observing that in a typical analysis an electric dipole is inferred on a moving magnetic dipole because, although the current density is Lorentz-transformed from the moving frame to the lab frame, the time and position coordinates of which the current density is a function are not. This error also ignores the relativity of simultaneity.

There is a subtlety here, however. As Furry[25] has shown, there is angular and linear momentum in the field of a system consisting of a point charge and an Amperian magnet, given in the rest frame (S') of the charge-magnet system of Mansuripur by, respectively,

$$\boldsymbol{L}' = \frac{\mu_o q \boldsymbol{\mu}'}{4\pi\gamma a} = \frac{\mu_o q \mu'}{4\pi\gamma a}\hat{\boldsymbol{k}} \quad \text{and} \quad \boldsymbol{P}' = \frac{1}{c^2}\boldsymbol{E}' \times \boldsymbol{\mu}' = -\frac{\mu_o q \mu'}{4\pi\gamma^2 a^2}\hat{\boldsymbol{\jmath}}, \tag{20}$$

where the angular momentum is taken about the center of the magnet. Here, q is the point charge at the origin of the S' coordinate system, $\boldsymbol{\mu}'$ is the magnetic moment of the magnetic dipole, the magnetic dipole is on the x' axis at $x' = \gamma a$ with its moment in the positive z' direction, and $\boldsymbol{E}'$ is the electric field at the location of the dipole.

The angular-momentum four-tensor in the S' frame has non-zero components given by

$$L_{z'} = L^{1'2'} = -L^{2'1'} = \frac{\mu_o q \mu'}{4\pi\gamma a} \quad \text{and} \quad L^{2'4'} = -L^{4'2'} = -ct' P_{y'}, \tag{21}$$

where $P_{y'}$ is the linear momentum in the y' direction,

$$P_{y'} = -\frac{\mu_o q \mu'}{4\pi\gamma^2 a^2}. \tag{22}$$

When this quantity appears as a *space-time component* in the angular momentum four-tensor, it depends on time. Transforming the tensor

$$L^{\mu'\nu'} = \begin{pmatrix} 0 & L_{z'} & 0 & 0 \\ -L_{z'} & 0 & 0 & -ct'P_{y'} \\ 0 & 0 & 0 & 0 \\ 0 & ct'P_{y'} & 0 & 0 \end{pmatrix} \tag{23}$$

to the S frame gives the component $L_z = L^{12}$ as

$$L_z = \gamma L^{1'2'} + \gamma \frac{v}{c} ct' P_{y'} = \gamma(L^{1'2'} + vt'P'). \tag{24}$$

The time derivative of this gives the time rate of change of the angular momentum, which is the torque involved. With $t = \gamma t'$, this torque is

$$\frac{dL_z}{dt} = vP' = -\frac{\mu_o v q \mu}{4\pi a^2}, \tag{25}$$

where $\mu = \gamma\mu'$.

This torque resides in the electromagnetic field. Where is the equal and opposite torque in S necessary to maintain conservation of angular momentum? There is no torque in the S' frame, so the torque must also be zero in the S frame. It turns out that the torque that counters this is the torque purported to be mechanical and to arise from the interaction of the point charge with the induced electric dipole in the moving frame.

However, I have shown[26] that the counter torque resides in the electromagnetic field and arises from the interaction of the charge q with the magnetic dipole and not with the non-existent electric dipole. The interaction of the electric field of the point charge with the current density in the magnet produces a force density *in the time slot* of the Lorentz force-density four-vector. This is an important observation as it implies the force density is in the field. The force density is given by[26]

$$f_{ct'} = \frac{J^{1'}E^{1'}}{c} + \frac{J^{2'}E^{2'}}{c}, \tag{26}$$

where $J^{\mu'} = \rho' u'(-sin\phi', cos\phi', 0, 0)$ is the current density in the x'-y' plane and $E^{\mu'}$, the electric field at the location of the dipole, has components given approximately by[26]

$$E^{1'} \approx \frac{q(\gamma a + R' cos\phi')}{4\pi\epsilon_o \gamma^3 a^3} \quad \text{and} \quad E^{2'} \approx \frac{qR' sin\phi'}{4\pi\epsilon_o \gamma^3 a^3}. \tag{27}$$

ρ' is the charge density of the current, u' is the drift speed, R' is the radius ($R' << a$) of the current loop, and ϕ' is the azimuth angle centered on the dipole. When the force-density four-vector is transformed to the S frame, it is

$$f^{\mu} = (\gamma \frac{v}{c} f_{ct'}, 0, 0, \gamma f_{ct'}), \tag{28}$$

where v is the speed of S in the x direction with respect to S'. It can be shown[26] that the net force in the S and S' frames is zero.

However, the torque is not zero. You can now calculate the torque in S' frame as

$$\begin{aligned} \tau^{2'4'} &= \int_{V'} (y' f_{ct'} - ct' f'_y) dV' = \int_{V'} y' f_{ct'} dV' = \frac{R'^2 \lambda' u'}{c} \int_0^{2\pi} \left(-\frac{q(\gamma a + R' cos\phi')}{4\pi\epsilon_o \gamma^3 a^3} \right) sin^2\phi' d\phi' \\ &= -\frac{\lambda'(u'/c) q \pi R'^2}{4\pi\epsilon_o \gamma^2 a^2}, \end{aligned} \tag{29}$$

where the substitution $\rho' dV' = \lambda' R' d\phi'$ has been made in the integral with λ' as the linear charge density of the charge carriers and $y' = R' sin\phi'$. This torque, *in a space-time slot*, gives rise to a torque about the z axis when transformed to the S frame as follows,

$$\begin{aligned} \tau_z &= \tau^{12} = \gamma \frac{v}{c} \tau^{4'2'} = \gamma \frac{v}{c} (-\tau^{2'4'}) \\ &= \frac{\mu_o v q \mu}{4\pi a^2}, \end{aligned} \tag{30}$$

where the magnetic moment in the S frame is $\mu = \gamma\mu' = \gamma I'\pi R'^2$, and $I' = \lambda' u'$. This torque is also in the electromagnetic field and cancels the torque in Eq. (25).

IV. AHARONOV-CASHER EFFECT

Working with an analogy to the Aharonov-Bohm effect where quantum interference occurs between charged particles traveling on either side of a solenoid, Aharanov and Casher proposed the same effect would be seen for magnetic neutral particles traveling on either side of, for example, a line of charge[7]. The proposed Aharonov-Casher (AC) effect included the proposition that neutrons would not experience a force while moving in an electric field. Neutrons passing either side of the line of charge with their magnetic moments parallel to the line and to each other would experience unequal phase shifts in their wave functions – what is called a "topological quantum effect" – resulting in a phase difference of

$$\Delta\phi = \mu_o \lambda \mu / \hbar, \tag{31}$$

where λ is the linear charge density, μ is the magnetic moment of the neutron, and $\hbar$ is the reduced Planck's constant. This would appear as a diffraction pattern in an experiment.

Boyer[10] disputed the notion that a neutron in an electric field would not experience a force. Instead, he argued that a moving neutron, modeled as an Amperian magnet, would sport an electric dipole $\boldsymbol{p}$ which would experience a force in an electric field $\boldsymbol{E}$ given by

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}. \tag{32}$$

With an electric field given by that due to a line of charge,

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_o r^2}\boldsymbol{r}, \tag{33}$$

where $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ measured from the line of charge, Boyer computed a force on the neutron given in SI units by

$$\boldsymbol{F} = \frac{\mu_o \mu \lambda v_o}{2\pi r^4}\left[(y^2 - x^2)\hat{\boldsymbol{\imath}} - 2xy\hat{\boldsymbol{\jmath}}\right]. \tag{34}$$

The force is that on a neutron with its magnetic moment parallel to the line of charge (in the positive z direction) moving in the positive y direction with speed v_o. The electric dipole in this case is given by $\boldsymbol{p} = \mu v_o / c^2 \hat{\boldsymbol{\imath}}$.

Boyer considers two such neutrons traveling in the positive y direction with speed v_o, one at $x = +a$ and one at $x = -a$. He assumes the paths will not vary much from straight lines, an assumption that seems justified considering that $\mu_o\mu/2\pi m = 1.15 \times 10^{-6}$ J·m/A·kg. He finds that a neutron passing on the positive x side of the wire is delayed with respect to one passing on the other side by an amount $\Delta y = \mu_o\mu\lambda/mv_o$, which results in the same phase shift as found by Aharonov and Casher in Eq. (31) given above. Hence Boyer claims the phase shift of the AC effect is due to classical lag rather than a quantum topological effect.

Arahonov *et al.*[11] responded that Boyer overlooked the effect of hidden momentum in the charge-magnet system. They equated the net force acting on the neutron (their equation (6)) to that acting between the line of charge and the induced electric dipole plus that due to the supposed rate of change of the hidden momentum and found that the net force on the neutron is zero, thus refuting Boyer's argument.

Mansuripur[27] argued that hidden momentum was not necessary to show the AC effect was not due to a classical lag, appealing to the use of the Einstein-Laub force[6] instead of the Lorentz force. However, he still assumed there was an electric dipole induced on the moving magnetic dipole. Since there is neither hidden momentum[12–15] in an charge-magnet system nor an electric dipole on a moving magnetic moment[22,23], the AC effect needs another look.

I have shown[14] that you have to take into account how a charge-magnet system is formed to understand its momentum; that is, by applying an electric field to a magnetic dipole or forming a magnetic dipole in a pre-existing electric field. When this is done mechanical momentum is imparted to both the source of the electric field and the magnet in equal amounts and in the same direction (unless the components are held statationary by an external agent).

When a point charge moves into the vicinity of a small Amperian magnet the magnet gains an amount of mechanical linear momentum[14] $\boldsymbol{\mu} \times \boldsymbol{E}/2c^2$, where the line between the charge and the dipole is perpendicular to the magnetic moment. An equal amount is gained by the charge. The momentum is the result of the operation of Lorentz forces. An amount of momentum opposite and equal in magnitude to that gained by both the charge and the magnet is deposited in the electromagnetic field. The force on the magnet is proportional to the displacement current at the location of the dipole, which is twice as large for a line of charge as for a point charge. Hence the momentum gained by the magnet in this case is

$$P_m = \boldsymbol{\mu} \times \boldsymbol{E}/c^2. \tag{35}$$

As the magnet moves through the field, the mechanical momentum will in general change with time as Lorentz forces act on the magnet. An equal change in momentum will occur to the line of charge (or to its support). An equal and opposite change in electromagnetic field momentum will occur at the same time. With the electric field given by Eq. (33), the force on the magnetic dipole given by dP_m/dt in x and y components is, respectively,

$$F_x = \frac{\mu_o \mu \lambda}{2\pi r^2}\left[\frac{2}{r^2}(\boldsymbol{r}\cdot\boldsymbol{v})y - \dot{y}\right] = m\ddot{x}, \tag{36}$$

and

$$F_y = \frac{\mu_o \mu \lambda}{2\pi r^2}\left[-\frac{2}{r^2}(\boldsymbol{r}\cdot\boldsymbol{v})x + \dot{x}\right] = m\ddot{y}. \tag{37}$$

Here $\boldsymbol{v}$ is the velocity of the magnet and m is the magnet's mass. When you make the same assumptions as Boyer (initial velocity $= \dot{y}\hat{\boldsymbol{j}} = v_o\hat{\boldsymbol{j}}$ and $x = \pm a$), these equations become

$$F_x = -\frac{\mu_o \mu \lambda v_o}{2\pi r^4}(y^2 - a^2) = m\ddot{x}, \tag{38}$$

and

$$F_y = \frac{\mu_o \mu \lambda v_o}{2\pi r^4}(2 \pm ay) = m\ddot{y}. \tag{39}$$

If you follow the same argument as Boyer, you would come up with an AC phase shift the same as was found by Aharonov and Casher and by Boyer.

To clarify this picture, you can recast the force equations in terms of polar coordinates. The force equation is then

$$\boldsymbol{F} = -\frac{\mu_o \mu \lambda}{2\pi r^2}(r\dot{\phi}\hat{\boldsymbol{r}} + \dot{r}\hat{\boldsymbol{\phi}}) = m[(\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}}]. \tag{40}$$

An obvious solution is when the magnetic dipole is stationary, achieved by using mechanical force to place the dipole at rest in the electric field. A more interesting solution occurs when r is constant. The equation reduces to

$$\frac{\mu_o \mu \lambda}{2\pi r^2}\hat{\boldsymbol{r}} = mr\dot{\phi}\hat{\boldsymbol{r}}. \tag{41}$$

According to this equation, the magnet can execute a circle in the counterclockwise direction (assuming λ is positive) with an angular speed of

$$\omega = \frac{\mu_o \mu \lambda}{2\pi m r^3}. \tag{42}$$

The circumference of the orbit where the speed of the dipole is $v = \omega r$ is

$$2\pi r = \frac{\mu_o \mu \lambda}{mv}. \tag{43}$$

This is the same as the lag found by Boyer and again confirms that the AC effect is due to a classical lag rather than a quantum effect.

V. CONCLUSIONS

Hidden momentum is not present in charge-magnet systems[12,14] and cannot be invoked to solve electromagnetic paradoxes. In the case of the paradox treated by Vaidman and others[3,16,17], there really is no paradox. Just because a Lorentz-transformed field is not observed in a particular reference frame, does not mean the interaction of the field with systems in the reference frame cannot be observed. Just as you can observe the Lorentz force on a charge moving in a magnetic field without being able to measure the Lorentz-transformed electric field acting on the charge, you can also observe the torque on a magnet moving in an electric field without being able to measure the Lorentz-transformed magnetic field.

In the case of the Mansuripur paradox[4], the lack of an electric dipole induced on a moving magnetic dipole means the magnetic dipole does not experience a torque as a result of an electric field acting on the non-existent electric dipole[22,23]. There is therefore no need to replace the Lorentz force with that of Einstein and Laub[6] in this case.

There are torques in the electromagnetic field of the frame in which the charge-magnet system of Mansuripur's paradox is moving. One is due to linear momentum in a space-time component of the angular momentum four-tensor in the rest frame of the system Lorentz-transformed to the moving frame. The other is due to the interaction of the charge with the electric current in the magnet in the system's rest frame, again due to a space-time component, this time in the torque four-tensor Lorentz-transformed to the moving frame. These are not mechanical torques and cancel out in the moving frame.

The analysis of Boyer[10] that the Aharonov-Casher effect is due to a classical lag appears to be correct in principle, except that the force on the magnet traveling through the electric field is not due to the electric field interacting with an induced electric dipole on the moving magnet, but rather is due to the changing mechanical momentum of the magnet (and an opposite change in the field momentum) as it moves through the electric field.

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