On Controversies in Electromagnetic Systems

F. R. Redfern[1]

1904 Corona Dr, Austin, TX 78723

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There have been controversies in the physics literature over questions concerning systems consisting of magnets and electric fields for over 100 years, and not all have been settled to everyone's satisfaction. In particular it has been accepted for over 50 years that Amperian magnets in electric fields contain hidden mechanical momentum and an electric dipole is induced on a moving magnetic dipole. The physics literature is filled with references to these claims; they show up in widely used textbooks and are invoked to solve electromagnetic paradoxes. However, I show that these claims violate special relativity and momentum conservation, and controversies are solved without them.

I. INTRODUCTION

It has been known for a long time that electromagnetic fields can contain both linear and angular momentum[1,2]. The linear electromagnetic momentum per unit volume in free space is given in SI units by

$$\boldsymbol{\wp}_{em} = \varepsilon_o \boldsymbol{E} \times \boldsymbol{B}, \tag{1}$$

where ε_o is the permittivity of free space, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic field. The quantity of linear electromagnetic field momentum in a region of space requires a volume integral of Eq. (1) over that region. The electromagnetic angular momentum density about a certain point involves taking the vector product between a position vector centered on that point and the linear momentum density at a point in space, that is,

$$\boldsymbol{l}_{em} = \varepsilon_o \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}). \tag{2}$$

Once again you must integrate over a volume of space to find the electromagnetic angular momentum in that space.

This article is concerned with linear and angular momentum of isolated electromagnetic systems in free space. There are a number of paradoxes associated with these systems that haven't been resolved to everyone's satisfaction. In what follows I will examine momentum in electromagnetic systems containing magnetic dipoles and show that the ideas of hidden momentum and induced electric dipoles on moving magnetic dipoles violate special relativity and momentum conservation. Controversies and paradoxes can be solved without these erroneous concepts.

II. MOMENTUM IN A CHARGE-MAGNET SYSTEM

A. An Unshielded Magnet

In 1969 Furry[3] calculated the electromagnetic linear and field angular momentum in charge-magnetic dipole systems for both an Amperian magnet where the magnetism is produced by electric current and a magnetic-pole model where the magnetism is produced by a pair of magnetic poles. In his calculation of linear field momentum involving an Amperian magnet, a magnetic dipole consisting of a uniformly magnetized sphere where the magnetization is due to a surface current was situated at the origin of a Cartesian coordinate system with its dipole moment $\boldsymbol{\mu}$ directed at an arbitrary angle in the *x*-*z* plane. (The specific magnet model was necessary to avoid a mathematical singularity at the location of the magnet.) A point charge q was located at $z = a$. He found the linear field momentum of a charge-Amperian magnet system to be

$$\boldsymbol{p}_{em} = \frac{\mu_o q}{4\pi a^3} \boldsymbol{\mu} \times \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{\mu}/c^2, \tag{3}$$

where the second expression on the right assumes the electric field is uniform over the extent of the magnetic dipole (small dipole approximation). If the magnet is due to a pair of isolated poles, he found the linear field momentum to be zero. He also found that any distribution of electric field and magnetic-pole pairs would have zero field linear momentum.

The field angular momentum about the location of magnet (this time with no need for a specific magnet model such as the uniformly magnetized sphere) was found to be

$$\boldsymbol{L}_{em} = \boldsymbol{a} \times \boldsymbol{p}_{em} = \frac{\mu_o q}{4\pi} \left[\frac{\boldsymbol{\mu} q}{a} - \frac{(\boldsymbol{\mu} \cdot \boldsymbol{a})\boldsymbol{a}}{a^3} \right]. \tag{4}$$

When the electric field is perpendicular to the magnetic moment, the field angular momentum is

$$\boldsymbol{L}_{em} = \frac{\mu_o q \boldsymbol{\mu}}{4\pi a}. \tag{5}$$

In contrast to the case of linear momentum, the field angular momentum was the same whether the magnetic dipole was Amperian or a magnet-pole pair.

The presence of linear and angular electromagnetic field momentum in a system consisting of a point charge and a magnetic dipole is hard to understand unless there is some way to balance the field momentum with mechanical momentum. This was what Shockley and James[4] set about to do in their 1967 paper where their model consisted of an Amperian magnet made of non-conducting material and two oppositely charged point particles equidistant on either side of the magnet and where the magnetic moment was perpendicular to the line between the two charges.

The two oppositely charged particles guaranteed that there would be no field angular momentum about the point where the magnet was located. With this model they only attempted to propose the presence of (hidden) mechanical linear momentum in the magnet equal and opposite to the linear field momentum.

However, they made an error in identifying mechanical momentum in their model by ignoring how it was assembled. I have shown[5] how linear mechanical momentum is imparted to the model of Shockley and James by bringing the two opposite charges in from a great distance to the vicinity of the magnetic dipole. This momentum is transferred to an external agent if that agent exerts mechanical forces to balance the Lorentz forces acting on the charges and magnet.

Here, I will bring a single charge in from a great distance to the vicinity of an Amperian magnetic dipole to show that the linear and angular mechanical momentum generated is equal and opposite to that found by Furry[3]. As the point charge q is moved toward the magnetic dipole, it experiences a Lorentz force, and the dipole will experience an equal Lorentz force. In this calculation both forces will be countered by an external agent such that the magnetic dipole remains stationary and the point charge moves in a straight line to its final position near the magnet. This is so that the charge and dipole end up in the same configuration as in Furry's calculation. The external agent will therefore be the recipient of the momentum generated by the Lorentz forces. The forces involved in assembling the system are equal and opposite so that the momentum parallel to the motion of the charge remains zero.

To keep the mathematics simple for the sake of argument, I will have the electric field perpendicular to the magnetic moment. The magnetic dipole will be at the origin of the coordinate system with its moment pointing in the positive z direction. The point charge will be moved slowly (to avoid any radiation from acceleration) from a great distance along the negative x axis in the positive direction to a point $x = -r$, where r is the distance between charge and magnet. The magnetic field at the location of the charge as it moves along the x axis is

$$\boldsymbol{B} = -\frac{\mu_o IA}{4\pi|x|^3}\hat{\boldsymbol{k}}, \tag{6}$$

such that the Lorentz force on the charge is

$$\boldsymbol{F}_L = qv\hat{\boldsymbol{\imath}} \times (-\frac{\mu_o IA}{4\pi|x|^3}\hat{\boldsymbol{k}}) = \frac{\mu_o qIAv}{4\pi|x|^3}\hat{\boldsymbol{\jmath}}, \tag{7}$$

where I is the current in the Amperian dipole, A is the area of the current loop, $v = dx/dt$ is the speed of the charge, and the magnetic moment is $\boldsymbol{\mu} = IA\hat{\boldsymbol{k}}$.

The impulse imparted to the charge as it moves along the x axis from $x = -\infty$ to $x = -r$ can be calculated as follows[5].

$$\boldsymbol{P}_q = \int_{-\infty}^{-r} \boldsymbol{F}_L dt = -\frac{\mu_o qIA}{4\pi}\hat{\boldsymbol{\jmath}} \int_{-\infty}^{-r} \frac{dx}{x^3} = \boldsymbol{\mu} \times \boldsymbol{E}/2c^2, \tag{8}$$

where $\boldsymbol{E}$ is the electric field at the location of the dipole in the small dipole approximation.

There has been some controversy over the correct formula for calculating the force on a magnetic dipole in a magnetic field[6–8], so I will calculate it from a more fundamental perspective. The result is the same as that found from the traditional formula, given by

$$\boldsymbol{F} = \nabla(\boldsymbol{\mu} \cdot \boldsymbol{B}), \tag{9}$$

the formula Franklin has argued is correct[6]. The moving charge q produces a magnetic field at the dipole. This is the magnetic field to be used in the above equation to calculate the force.

The force can be calculated from the magnetic field produced by the displacement current acting on the charge current of the magnet (Figure 1). In the slow-motion approximation where,

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1, \tag{10}$$

the displacement due to q at the magnetic dipole is

$$\boldsymbol{D} = \frac{1}{4\pi}\frac{q}{r^2}\hat{\boldsymbol{i}}. \tag{11}$$

The displacement current density is equal to the time rate of change of the displacement, $\boldsymbol{J}_D = d\boldsymbol{D}/dt$. Integrating the inner product of the displacement current density with an area gives the displacement current through that area. Then Ampere's law can be used to calculate the magnetic field around the area in situations sufficiently symmetrical.

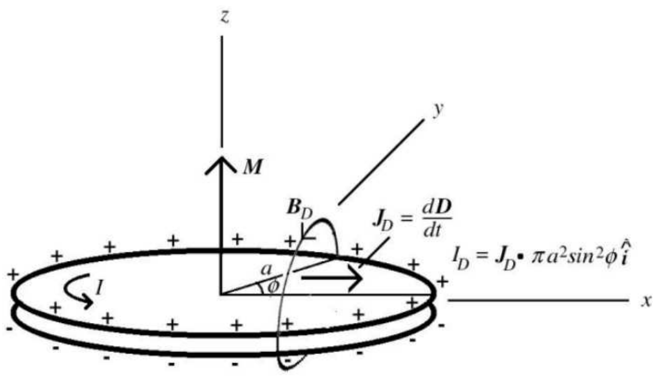

FIG. 1. Displacement Current and Magnetic Field at Magnet Model

In this case you have circular magnetic field lines centered on the x axis and oriented in a counterclockwise direction with respect to the positive x direction. The strength of the field of a magnetic loop will depend on the displacement current within it. The magnetic field loops acting on the charge current will have cross-sectional areas defined by circles with diameters equal to the distance between the edges of the disks parallel to y (Figure 1). The area of a given loop is $\pi a^2 sin^2\phi$, where a is the radius of the disk and ϕ is the usual azimuth angle of a spherical coordinate system. The displacement current as a function of ϕ involved in the interaction is therefore

$$I_D = \frac{d\boldsymbol{D}}{dt} \cdot \hat{\boldsymbol{i}}\pi a^2 sin^2\phi = \frac{qva^2}{2r^3} sin^2\phi, \tag{12}$$

where $\boldsymbol{v} = (-dr/dt)\hat{\boldsymbol{\imath}}$ when you take the time derivative of $\boldsymbol{D}$. From the integral form of Ampere's law, you find the magnetic field at the rims of the disks as a function of ϕ,

$$\boldsymbol{B}_D = \frac{\mu_o I_D}{4\pi a sin\phi}\hat{\boldsymbol{k}} = \frac{\mu_o q a v}{4\pi r^3} sin\phi \hat{\boldsymbol{k}}. \tag{13}$$

Now you integrate the Lorentz force around the current loop to get the total force due to the magnetic field of the displacement current.

$$\boldsymbol{F}_D = I\oint d\boldsymbol{l} \times \boldsymbol{B}_D = I\oint a d\phi \hat{\boldsymbol{\phi}} \times \boldsymbol{B}_D = \frac{\mu_o q I A v}{4\pi r^3}\hat{\boldsymbol{j}}, \tag{14}$$

where $\hat{\boldsymbol{\phi}} = -sin\phi\hat{\boldsymbol{\imath}} + cos\phi\hat{\boldsymbol{j}}$. To get the impulse on the dipole, you integrate over time. (The integral becomes one taken over the distance r, since $v = -dr/dt$.)

$$\boldsymbol{P}_{dp} = \frac{\mu_o q I A}{8\pi r^2}\hat{\boldsymbol{j}} = \boldsymbol{\mu} \times \boldsymbol{E}/2c^2. \tag{15}$$

Note that this is equal to the impulse applied to the charge, Eq. (8), both in magnitude and direction, and when the impulses are added together you get a result that is equal and opposite the linear field momentum of Furry, Eq. (3). So, you have started out with components with zero momentum and end up with a charge-magnet-external agent system with zero linear momentum.

The next task is to calculate the mechanical angular momentum the charge-dipole system acquires as the point charge is brought in from a great distance. The external agent applying mechanical forces will again be the final recipient of the mechanical angular momentum.

As the point charge q is moved toward the magnetic dipole, it experiences the Lorentz force given by Eq. (7), and the dipole will experience an equal force. As in the calculation of Furry, the angular momentum will be taken about the location of the magnetic dipole. The force of the external agent, of course, produces no angular momentum about the location of the dipole. The mechanical angular momentum acquired by the external agent is given by

$$\begin{aligned} \boldsymbol{L}_{mech} &= \int \boldsymbol{r} \times \boldsymbol{F}_L dt = -\frac{\mu_o q \mu \hat{\boldsymbol{k}}}{4\pi}\int_{-\infty}^{-r}\frac{dx}{x^2} \\ &= -\frac{\mu_o q \boldsymbol{\mu}}{4\pi r}. \end{aligned} \tag{16}$$

This mechanical angular momentum is equal and opposite to the field angular momentum found by Furry Eq. (5). No hidden momentum in the magnet, either linear or angular, is necessary for momentum conservation. In fact, the presence of hidden momentum would violate momentum conservation. Although there could be Coulomb forces on the charge and magnet due to induced charges on the magnet, these would not contribute to the momentum of the system.

B. A Shielded Magnet

For this calculation I will use some previous results where a magnet is formed in a uniform electric field[5]. To avoid the problems raised by an unbounded uniform electric field in this reference, I formed the magnet inside a non-conducting spherical shell with a surface charge density that produced a uniform field inside the shell and a dipolar field externally. It was shown that linear momentum was conserved without the need for hidden momentum, since the mechanical momentum of the shell was equal and opposite to the field momentum. There was no angular momentum in this model.

In the present calculation, the shell will be conducting and a current will be introduced in the shell by an external agent. Because of the absence of an electric field inside the shell there is no question of hidden momentum. Lenz' law guarantees that the net mechanical angular momentum of the charge carriers and the external agent will be zero. Rather than bringing in a positive charge from a great distance, here there will be a positive charge q a distance r in the negative x direction from the center of the magnet and held stationary by mechanical forces produced by an external agent as the magnet is formed slowly by steadily increasing the current in the counterclockwise direction about the z axis. The growing magnetic moment will point in the z direction.

The conducting shell has a radius R and the two-dimensional charge density of the charge carriers, σ, is uniform across the shell. The two-dimensional current density will have a magnitude

$$K = \sigma\omega R sin\theta, \tag{17}$$

where ω is the angular speed of the charge density, and θ is the polar angle. The magnetic moment will be in the positive z direction:

$$\boldsymbol{\mu} = \frac{4}{3}\pi\sigma\omega R^4\hat{\boldsymbol{k}}, \tag{18}$$

meaning the current density is given by[3]

$$\boldsymbol{K} = \frac{3}{4\pi R^4}\boldsymbol{\mu}\times\boldsymbol{R}. \tag{19}$$

The induced charge density on the conducting shell produces an electric field inside the shell that cancels that due to the point charge. This field can be assumed to be uniform in the small-dipole approximation. The growing magnetic field creates an emf that acts on the induced charge such that a linear momentum of

$$\boldsymbol{P}_{shell} = -\frac{\boldsymbol{\mu}\times\boldsymbol{E}}{c^2}, \tag{20}$$

is imparted to the shell[5]. (The sign difference between this and the referenced material is due the difference in the direction of $\boldsymbol{E}$.)

There will also be an emf acting on the point charge q. The magnetic flux inside the radius r will be

$$\Phi_{<r} = \frac{\mu_o \dot{\mu} t}{2r}, \tag{21}$$

where $\dot{\mu} = d\mu/dt$. The growth of the flux inside the radius r will establish an emf $\mathscr{E} = d\Phi_{<r}/dt = \mu_o \dot{\mu}/2r$ at the radius r in the clockwise direction about the z axis. The force on q will then be

$$\boldsymbol{F}_q = q\frac{\mathscr{E}}{2\pi r}\hat{\boldsymbol{j}} = \frac{\mu_o q \dot{\mu}}{4\pi r^2}\hat{\boldsymbol{j}}. \tag{22}$$

Without an external agent holding q stationary with mechanical forces, the charge would accelerate. (Or, you could have the mass of q to be extremely large.) In the former case the external agent would receive an impulse of

$$\boldsymbol{P}_q = \int \boldsymbol{F}_q dt = \frac{\mu_o q \mu}{4\pi r^2}\hat{\boldsymbol{j}} = \frac{\boldsymbol{\mu} \times \boldsymbol{E}}{c^2}, \tag{23}$$

where $\boldsymbol{\mu}$ is the final value of the magnetic moment and $\boldsymbol{E}$ is the electric field at the location of the magnet in the small-dipole approximation. Note that this is the negative of Eq. 20, such that there is no net linear mechanical momentum in the system.

What about linear field momentum? Furry's result[3] for the linear field momentum without the magnet model he used to avoid a mathematical singularity was, in SI units,

$$\boldsymbol{P}_F = -\frac{\boldsymbol{\mu} \times \boldsymbol{E}}{3c^2}, \tag{24}$$

his equation (30), where $\boldsymbol{\mu}$ and $\boldsymbol{E}$ are defined as before. This is the appropriate equation here for the linear momentum due to the electric field of q and the the magnetic field of $\boldsymbol{\mu}$, since the electric field of q does not penetrate the shielded magnet. The linear field momentum due to the dipolar field of $\boldsymbol{\mu}$ and that of the charge of the shell was calculated previously[5], and for this situation is given by

$$\boldsymbol{P}_{dd} = \frac{\boldsymbol{\mu} \times \boldsymbol{E}}{3c^2}. \tag{25}$$

So the net linear field momentum is also zero and liner momentum has been conserved in the assembly of a charge in the vicinity of a shielded magnet.

The analysis for angular field momentum is just as was given by Furry[3], since the magnet is assumed to be so small that the angular field momentum in its immediate vicinity is negligible. For the same reason the result for the mechanical momentum given for the unshielded magnet given above also holds. Hence angular momentum is also conserved.

III. THE MANSURIPUR PARADOX AND CHARGE SEPARATION ON MOVING MAGNETIC DIPOLES

A news article that appeared in the 27 April 2012 issue of the journal *Science*[9] reviewed a claim that provoked a lot of discussion among many researchers involved in electromagnetic theory and special relativity. It was claimed by Mansuripur[10] that the Lorentz force of electromagnetism was not compatible with special relativity. His argument was based on a paradox involving angular momentum in a charge-magnet system.

The paradox is depicted in Figure 2. In the inertial reference frame S', observer O' is stationary with respect to a positive charge q at the origin and a magnetic dipole $\boldsymbol{\mu}' = \mu'\hat{\boldsymbol{k}}$ at $x' = a$. She sees no reason for there to be any interaction between the charge and the dipole. However, frame S' is moving to the right in the inertial frame of observer O, and, according to the conventional idea, he should see an electric dipole on the magnetic dipole, indicated by the charge symbols in Figure 2. In his frame of reference, he should see the positive side of the magnetic dipole repelled by charge q and the negative side attracted resulting in a torque acting on the magnetic dipole – a torque which is not observed by O'. This situation is clearly outlawed by the principle of relativity (not to mention common sense), so Mansuripur claimed the Lorentz force responsible for the torque is not in tune with relativistic principles and should be replaced by another force, the one proposed by Einstein and Laub[11].

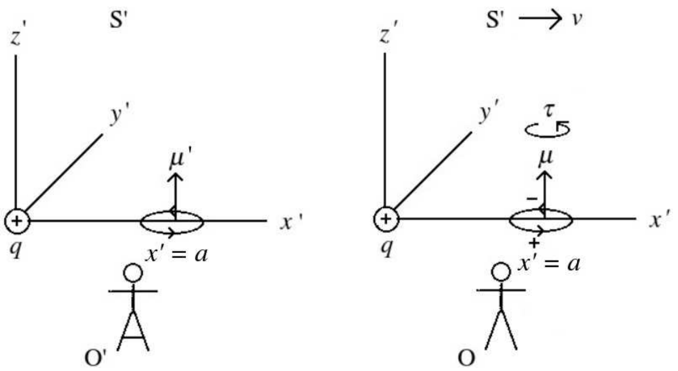

FIG. 2. The Mansuripur Paradox

A resolution to the paradox preserving the Lorentz force based on hidden momentum has been proposed. (See, for example, Griffiths and Hnizdo[12].) However, the simplest resolution involves the proposal that an electric dipole is not present on a moving magnetic dipole[13,14]). With no electric charge separation on the magnetic dipole, there is no *mechanical* torque in either the S or

S' frames.

It is interesting that the belief there is an electric dipole on a moving magnetic dipole arises from a misapplication of relativity. When current density is Lorentz-transformed from an inertial reference frame in which there is no charge density to a frame moving with respect to that frame, the general result appears to be a charge density in the moving frame. The charge density four-vector in the S' frame is

$$j^{\mu'} = (j^{1'}, j^{2'}, j^{3'}, c\rho'), \tag{26}$$

where $j^{i'}$ is the current density in the x', y', and z' directions and ρ' is the charge density (in the time component of the four-vector). A Lorentz transformation to the S frame in which the S' frame is moving in the x direction with speed v is[15]

$$j^{\mu} = \gamma(j^{1'} - v\rho', j^{2'}, j^{3'}, c\rho' - vj^{1'}/c). \tag{27}$$

According to the above equation, there is a charge density in the S frame given by $-\gamma v j^{1'}/c^2$ even when ρ' is zero. This implies that the observer O sees the near side of the current loop in Figure 2 to be positive and the far side to be negative, implying he should see the result of a torque acting on the loop due to interaction with charge q.

Franklin[13] has provided a direct explanation as to why this is not true. He shows that Eq. (27) is not correct as naìvely interpreted because the current density is a function of the space and time coordinates which have to be transformed along with the current density itself. By not performing this transformation, the relativity of simultaneity is violated.

To see this is the case, consider Figure 3, which depicts a rectangular current loop in its rest frame, S'. The left-right length of the loop is l' and it carries a counterclockwise positive current I'. In the center of the bottom segment of the loop is a pulse generator that emits brief light pulses to the left and right simultaneously.

In the S' frame the pulses arrive at the same time at the detectors, but that is not the case in the lab (S) frame in which the loop is moving to the right with speed v. In the S frame, due to the fact the speed of light is the same in all frames, the left-moving pulse arrives at the detector at the trailing end of the loop at a time $\gamma v l'/c^2$ before the right-moving pulse arrives at the detector at the leading end of the loop. This is nothing more than the proverbial "train paradox" of relativity.

It can be shown that in the S frame an amount of charge $(vl'/c^2)I'$ passes the left side of the loop after the pulse arrives there and before the pulse arrives at the right side of the loop[14]. A similar situation with an opposite amount of charge occurs on the top segment of the loop, making

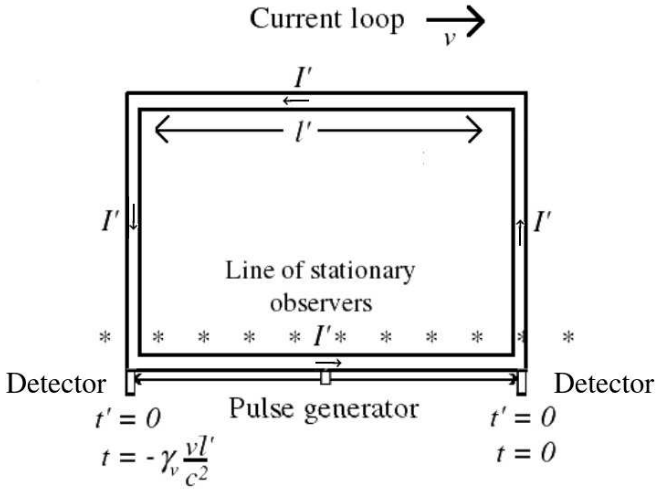

FIG. 3. A Moving Current Loop

the loop appear to have a charge separation. The amount of charge on the top and bottom loop segments determined between the times of the pulse arrivals is different in the S frame from that in the S' frame. The apparent charge separation in the S frame is just the amount necessary for an electric dipole to appear on the loop[14]. However, it is clear this charge is only due to the relativity of simultaneity – due to the fact that the pulse arrivals are not simultaneous in the S frame as they are in the S' frame.

One must also apply the argument of Franklin[13] to the magnetization-polarization tensor. The Lorentz-transformation of the tensor is not properly done without a concomitant transformation of the coordinates of the tensor components. When this is done no electric dipole appears on moving magnetic material, although there is, of course, an electric field generated according to Faraday's law. This error is unfortunately found in the classic text of Panofsky and Phillips[15].

IV. TORQUE IN A MOVING CHARGE-MAGNETIC DIPOLE SYSTEM

One area of possible confusion over the existence or non-existence of torque in a moving charge-magnetic dipole system, such as that which arises in the Mansuripur paradox, could be the fact that there is indeed torque in the system. The system contains linear and angular momentum in its electromagnetic field and both appear in the angular momentum four-tensor. When the system is moving, the linear momentum gives rise to a time-dependent angular momentum creating torque in the electromagnetic field.

Also, there is indeed torque due to the interaction between the charge and the magnetic dipole when the system is moving so long as the magnet is not shielded from the electric field, but this torque is also in the electromagnetic field and is not mechanical torque. These two torques are equal and opposite, canceling out with the result the angular momentum of the charge-magnet system is conserved with no mechanical or net electromagnetic torque present. These torques can be identified in a fully relativistic analysis.

In the following equation for the angular momentum four-tensor, $\boldsymbol{L}$ is the angular momentum of the system, $\boldsymbol{p}$ its linear momentum, m is the system mass, (x,y,z) is the position of the center of mass with respect to the point about which the angular momentum is taken, and t is the time in the rest frame of the system.

$$L^{\mu\nu} = \begin{pmatrix} 0 & L_z & -L_y & mcx - ctp_x \\ -L_z & 0 & L_x & mcy - ctp_y \\ L_y & -L_x & 0 & mcz - ctp_z \\ -mcx + ctp_x & -mcy + ctp_y & -mcz + ctp_z & 0 \end{pmatrix}.$$

Consider the charge-magnet system of the Mansuripur paradox. The angular momentum four-tensor for the electromagnetic field in the S'frame is,

$$L^{\mu'\nu'} = \begin{pmatrix} 0 & \frac{\mu_o q \mu}{4\pi a} & 0 & 0 \\ -\frac{\mu_o q \mu}{4\pi a} & 0 & 0 & \frac{\mu_o q \mu ct}{4\pi a^2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu_o q \mu ct}{4\pi a^2} & 0 & 0 \end{pmatrix}, \tag{28}$$

where Eqs. (3) and (4) have been used. Transforming this tensor to the lab frame S in which frame S' is moving with speed v in the positive x direction, you get, in the slow-motion approximation,

$$L^{\mu\nu} = \begin{pmatrix} 0 & \frac{\mu_o q \mu}{4\pi a} - \frac{\mu_o q \mu vt}{4\pi a^2} & 0 & 0 \\ -\frac{\mu_o q \mu}{4\pi a} + \frac{\mu_o q \mu vt}{4\pi a^2} & 0 & 0 & \frac{\mu_o q \mu ct}{4\pi a^2} + \frac{\mu_o q v \mu}{4\pi ca} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu_o q \mu ct}{4\pi a^2} - \frac{\mu_o q v \mu}{4\pi ca} & 0 & 0 \end{pmatrix}, \tag{29}$$

The z component of the angular momentum in the lab frame, $L_z = L^{12}$ is

$$L_z = \frac{\mu_o q \mu}{4\pi a} - \frac{\mu_o q \mu vt}{4\pi a^2}. \tag{30}$$

This angular momentum must be in the electromagnetic field since the first term on the right certainly is and the second is found from a Lorentz transformation of a linear momentum in the electromagnetic field. The time derivative of this gives the time rate of change of the angular momentum, which is the torque involved.

$$\frac{dL_z}{dt} = -\frac{\mu_o v q \mu}{4\pi a^2}. \tag{31}$$

As for the interaction between the unshielded current loop and the point charge q, look at the force four-vector on the current loop in the S' frame. Use the slow-motion approximation such that the current loop is centered on $x = x' = a$ at $t = t' = 0$ in both frames and where quantities unchanged between S and S' in this approximation are not primed. If the radius of the current loop is R, the electric field at a point on the loop $x = a + Rcos\phi$ and $y = Rsin\phi$ due to the charge q, where ϕ is the local azimuth angle measured in the positive direction from the x axis, is given by

$$\boldsymbol{E}' = \frac{1}{4\pi\varepsilon_o}\frac{q(\boldsymbol{a}+\boldsymbol{R})}{(a^2+R^2+2aRcos\phi)^{3/2}}, \tag{32}$$

where $\boldsymbol{a} = a\hat{\boldsymbol{\imath}}$ and $\boldsymbol{R} = R(cos\phi\,\hat{\boldsymbol{\imath}} + sin\phi\,\hat{\boldsymbol{\jmath}})$. The loop carries a current density given by

$$J^{\mu'} = \rho u'(-sin\phi, cos\phi, 0, 0), \tag{33}$$

where,

$$J_{x'} = -\rho u' sin\phi \quad \text{and} \quad J_{y'} = \rho u' cos\phi, \tag{34}$$

and where ρ is the charge density of the current and u' is the drift speed. Breaking up the electric field into x and y components (no z component is present at the loop) and applying the Lorentz electromagnetic field tensor, you get

$$E^{\mu'\nu'}J_{\nu'} = \begin{pmatrix} 0 & 0 & 0 & \frac{E_{x'}}{c} \\ 0 & 0 & 0 & \frac{E_{y'}}{c} \\ 0 & 0 & 0 & 0 \\ -\frac{E_{x'}}{c} & -\frac{E_{y'}}{c} & 0 & 0 \end{pmatrix} \begin{pmatrix} -J_{x'} \\ -J_{y'} \\ 0 \\ 0 \end{pmatrix}, \tag{35}$$

so

$$E^{\mu'\nu'}J_{\nu'} = \begin{pmatrix} 0 \\ \\ 0 \\ \\ 0 \\ \\ \frac{J_{x'}E_{x'}}{c} + \frac{J_{y'}E_{y'}}{c} \end{pmatrix}. \tag{36}$$

The force-density in the time slot is seen to be

$$f_{ct'} = \frac{J_{x'}E_{x'}}{c} + \frac{J_{y'}E_{y'}}{c}, \tag{37}$$

the same as the power density divided by c. This component is present because there is force on the current density due to the charge q, and the current density is moving in the lab frame.

Assuming the distance a is much greater than the loop radius R, the electric field components on the loop in S' are approximately (Eq. (32))

$$E_{x'} \approx \frac{q(a + Rcos\phi)}{4\pi\varepsilon_o a^3} \quad \text{and} \quad E_{y'} \approx \frac{qRsin\phi}{4\pi\varepsilon_o a^3}. \tag{38}$$

When you substitute $E_{x'}$ and $E_{y'}$ from the above equations and $J_{x'}$ and $J_{y'}$ from Eq. (34) into Eq. (37) and integrate over the volume, you find that the total four-force on the loop in S' is zero due to the angular dependence on ϕ. A four-vector that is zero in one frame of reference has to be zero in all other inertial reference frames, including, of course, the lab frame.

Nevertheless this force is responsible for the appearance of a torque in the lab frame, but this torque results from a force density in the time component of the four-vector rather than a space component, which implies it is not a mechanical torque but one confined to the electromagnetic field. The components of the antisymmetric torque four-tensor, given by the volume integral

$$\tau^{\alpha\beta} = \int_V (x^\alpha f^\beta - x^\beta f^\alpha) dV, \tag{39}$$

in S' acting on the current loop are not all zero. The volume integrals of the torque density that are zero are due to the ϕ dependence and the fact that $z = 0$. The non-zero pair (symmetric-antisymmetric partners) are $\tau^{2'4'}$ and $\tau^{4'2'} = -\tau^{2'4'}$. The calculation of $\tau^{2'4'}$ is carried out as follows, taking the origin about the center of the loop for the volume integration of the torque density,

$$\tau^{2'4'} = \int_{V'} (y' f_{ct'} - ct' f'_y) dV' = \int_{V'} y' f_{ct'} dV' \tag{40}$$
$$= \int_{V'} (Rsin\phi) \left(\frac{J_{x'} E_{x'}}{c} + \frac{J_{y'} E_{y'}}{c} \right) dV'.$$

To perform the volume integration, you assume that the wire of the loop is one-dimensional, which lets you make the substitution $\rho dV' = \lambda R d\phi$ where λ is the linear charge density of the charge carriers responsible for the current. This allows you to write the integral as

$$\tau^{2'4'} = \frac{R^2 \lambda u'}{c} \int_0^{2\pi} (-E_{x'} sin^2\phi + E_{y'} sin\phi cos\phi) d\phi. \tag{41}$$

The second integrand gives zero when integrated over ϕ. The first integrand gives

$$\tau^{2'4'} = \frac{R^2 \lambda u'}{c} \int_0^{2\pi} \left(-\frac{q(a + Rcos\phi)}{4\pi\varepsilon_o a^3} \right) sin^2\phi d\phi \tag{42}$$
$$= -\frac{q\mu/c}{4\pi\varepsilon_o a^2},$$

where $\mu = I'\pi R^2 = \lambda u' \pi R^2$. This torque, when transformed to the S frame, gives rise to a torque about the z axis, as follows (where $I = I'$ since $u = u'$ in the slow-motion approximation as it is the speed difference between charge carriers and background ions),

$$\tau_z = \tau^{12} = \frac{v}{c} \tau^{4'2'} = \frac{v}{c}(-\tau^{2'4'}) = \frac{\mu_o v q \mu}{4\pi a^2}. \tag{43}$$

This is the torque that is supposed to be mechanical in nature and produced by the interaction between the charge q and the presumed electric dipole on the magnetic dipole, both in the moving S' frame. However, this torque is actually in the electromagnetic field, not mechanical, and offsets the torque given in Eq. (31).

In the case of a shielded current loop, there can be no interaction between the current in the loop and the electric field of the charge. However, note that the entire system in this model includes the external agent with a mechanical linear and angular momentum equal and opposite to that of the electromagnetic field. Thus the total momentum is zero in both the S and S' frames. (A four-vector or four-tensor that is zero in one inertial frame is zero in all others.)

It is interesting to note that when the magnet dipole consists of a pair of magnetic poles, there is no linear field momentum but the field angular momentum is still present according to Furry[3]. With no linear momentum there is no time-dependent field angular momentum in the transformed angular momentum four-tensor and thus no field torque. Also, since there is no current density in the magnetic dipole, there is no interaction between the charge and the magnetic dipole when they are in motion and thus no field torque here either.

V. A CURRENT LOOP MOVING IN A UNIFORM ELECTRIC FIELD

In this paradox you have a uniform electric field ($\boldsymbol{E} = E\hat{\boldsymbol{k}}$) directed parallel to the positive z axis and a current loop initially moving in the positive x direction in the lab frame (S) at speed v with its magnetic dipole $\boldsymbol{\mu} = \mu\hat{\boldsymbol{\imath}}$ pointed in the direction of motion. This paradox has been treated by Bedford and Krumm[16], by Namias[17]), by Vaidman[7], and by Franklin[13]. In the rest frame of the loop the electric field is moving in the negative x direction with speed v. Hence there is a Lorentz-transformed magnetic field present at the loop (Figure 4).

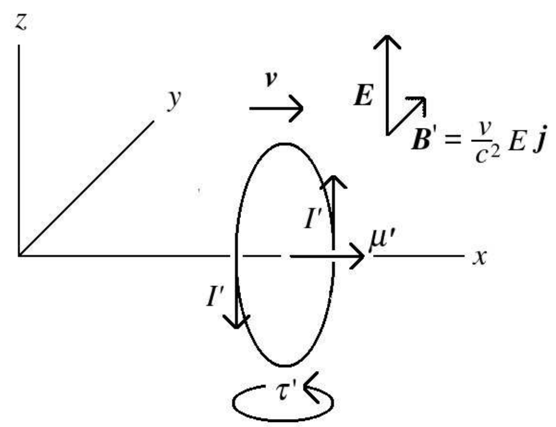

FIG. 4. The “Paradox” Treated by Vaidman

$$\boldsymbol{B}' = -\gamma \boldsymbol{v} \times \boldsymbol{E}/c^2 = \gamma \frac{v}{c^2} E\hat{\boldsymbol{\jmath}} \tag{44}$$

There is also a transformed electric field given by

$$\boldsymbol{E}' = \gamma \boldsymbol{E} = \gamma E\hat{\boldsymbol{k}}, \tag{45}$$

The presence of the magnetic field in the S' frame implies there is a torque on the current loop in that frame (so long as the electric field is not screened) given by

$$\boldsymbol{\tau}' = \boldsymbol{\mu}' \times \boldsymbol{B}' = \mu \frac{v}{c^2} E\hat{\boldsymbol{k}}, \tag{46}$$

where the term on the far right uses the slow-motion approximation; that is, $\gamma = 1$ ($v << c$), meaning $\boldsymbol{\mu}' = \boldsymbol{\mu}$ and $\boldsymbol{E}' = \boldsymbol{E}$. This approximation will be employed for the rest of this section.

The problem is there is no magnetic field in the lab frame and therefore (presumably) no torque. The (net) Lorentz force on the loop is zero. Why is it that an observer in S' records a torque that is not observed in the lab frame? Vaidman found a resolution for three versions of a magnetic dipole; however, he also claims that two of the dipole models contain hidden momentum. It is possible to resolve the paradox in a general way and show that hidden momentum, if it exists, spoils the resolution.

The hidden linear momentum in the two current loop models in the S' frame is given by Vaidman as[7]

$$\boldsymbol{P}_{hidden} = -\frac{1}{c}\int \phi \boldsymbol{J} dV = \boldsymbol{\mu} \times \boldsymbol{E}/c^2 = -\mu E/c^2 \hat{\boldsymbol{j}}, \tag{47}$$

when the magnetic dipole is parallel to the positive x direction. $\boldsymbol{J}$ is the current density in the loop and $\phi = -zE$ is the electric potential. (According to the above equation, the hidden momentum will change direction if $\boldsymbol{\mu}$ rotates.) If there is a time-dependent angular momentum $L_{z'}$ along the z axis about the center of the loop at time t ($= t'$ for the slow-motion approximation) and also hidden linear momentum in the loop given at that instant by Eq. (47), the angular momentum four-tensor (Eq. (28)) taken about the center of the loop in S' is

$$L^{\mu\nu} = \begin{pmatrix} 0 & L_{z'} & 0 & 0 \\ -L_{z'} & 0 & 0 & \mu E t/c \\ 0 & 0 & 0 & 0 \\ 0 & -\mu E t/c & 0 & 0 \end{pmatrix}. \tag{48}$$

(There is also linear momentum in the electromagnetic field which would be included in the complete tensor, but since the interaction between matter and the field is negotiated by the Lorentz force, there is no reason to include it when just considering the current loop.)

The torque on the loop is the time rate of change of its angular momentum in the rest frame (S') of the loop (dL_z/dt). When the angular momentum four-tensor is Lorentz-transformed to the S frame, the four-tensor will contain

$$L_z = L_{z'} - (v/c^2)\mu E t \tag{49}$$

in the x-y slot. The time derivative of this equation gives

$$dL_z/dt = dL_{z'}/dt - (v/c^2)\mu E, \tag{50}$$

such that the torque in the S frame does not equal that in the S' frame. So, we are back to the paradox of observers in different frames measuring different torques.

The fact is no hidden momentum exists in the current loop in S' as I have shown above. The torque acting on a current loop is that found in Eq. (46), unless the ambient electric field is shielded from the current by conducting material. Then $\boldsymbol{B}' = 0$ at the current and there will be no torque, something previously pointed out by Franklin[6]. If there is induced charge on the loop due to a lack of total shielding from the ambient electric field, the Lorentz force on the two charged sides of the loop will be parallel to the z direction and equal and opposite, canceling out and producing no torque but creating stress in the magnet.

Namias[17] and Vaidman[7] claim the there is a magnetic field inside the conducting loop due to the scalar potential that the charges induced by the ambient electric field to counter the ambient potential. This induced potential is supposed to give rise to a magnetic field in the S frame courtesy of the vector potential of the transformed induced potential. However, it appears to me the total scalar potential inside the conductor in both frames should be constant, and therefore there should be no transformed magnetic field in S. (See also Franklin[6].)

For a magnetic dipole like that of Shockley and James[4], there is no conducting material and the full torque of Eq. (46) should be realized. (Any induced charge will be stationary in S' frame.) Since there is no magnetic field in the lab frame in which the loop is moving, how is it that no torque is observed? Actually, the torque is observed as I will now explain.

It is correct that in the S frame the (net) Lorentz force on the magnetic dipole is zero, but that does not mean the torque is zero. In the S' frame there is a force acting on the positive y' side of the loop in the negative x' direction. On the negative y' side of the loop a force of the same magnitude is acting on the loop in the positive x' direction. These forces sum to zero and so the net force on the loop in both the S and S' frames is zero. When these two forces are transformed to the S frame, each is reduced by a factor of $1/\gamma$, but they are still there. The mass and thus the rotational inertia of the loop is increased by a factor of γ. The torque is reduced and the inertia is increased by the same factor such that the rotational motion is the same in both frames.

This argument brings up another point. It should be clear that you cannot create an interaction in a system where there is none by merely performing a Lorentz transformation. Neither can you Lorentz-transform away an interaction in a system. The interaction involving a Lorentz-transformed magnetic field and a magnet in its rest frame cannot be transformed away by observing the system in a frame moving with respect to the rest frame of the magnet, even if a Lorentz-transformed magnetic field does not exist in that frame.

My first objection to the Mansuripur paradox was in fact applying the point mentioned above.

A charge separation on a current-carrying wire or current loop is an interaction between opposite charges that cannot be produced by merely applying a Lorentz transformation from the rest frame of the system to a moving frame of reference.

VI. AHARONOV-CASHER EFFECT

The Aharonov-Bohm effect[18,19] was a surprising manifestation of the influence of the vector potential of electrodynamics on the quantum behavior of particles not subject to either a magnetic or electric field. It was predicted that two identical charged particles, passing either side of a solenoid would exhibit a phase difference in their wave functions, leading to a detectable interference pattern when the particles interacted after passing the solenoid. This was called a "topological quantum effect".

This was puzzling since there is no electric or magnetic field acting on the particles, and the vector potential field through which the particles traveled was thought by many to be only a mathematical convenience for working out electromagnetic problems. The electric and magnetic fields can be computed from the scalar and vector potentials of electromagnetism, but neither is unique. They can be transformed by gauge transformations and yet yield the same electric and magnetic fields. So it was a surprise that a field that was not considered exactly real could have real effects. (This was seen as a case of gauge invariance.)

Working with an analogy to the Aharonov-Bohm effect, Aharanov and Casher proposed the same effect would be seen for neutral magnetic particles traveling on either side of, for example, a line of charge[20]. The proposed Aharonov-Casher (AC) effect included the proposition that neutrons would not experience a force while moving in an electric field. Neutrons passing either side of the line of charge with their magnetic moments parallel to the line and to each other would experience unequal phase shifts in their wave functions resulting in a phase difference of

$$\Delta\phi = \mu_o \lambda \mu / \hbar, \tag{51}$$

where λ is the linear charge density, μ is the magnetic moment of the neutron, and $\hbar$ is the reduced Planck's constant. This could appear as a diffraction pattern in an experiment.

Boyer[21] disputed the notion that a neutron in an electric field would not experience a force. Instead, he argued that a moving neutron, modeled as an Amperian magnet, would sport an electric dipole $\boldsymbol{p}$ which would experience a force in an electric field $\boldsymbol{E}$ given by

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}. \tag{52}$$

With an electric field produced by a line of charge,

$$\boldsymbol{E} = \frac{\lambda}{2\pi\varepsilon_o r^2}\boldsymbol{r}, \tag{53}$$

where $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ is measured from the line of charge, Boyer computed a force on the neutron given in SI units by

$$\boldsymbol{F} = \frac{\mu_o \mu \lambda v_o}{2\pi r^4}\left[(y^2 - x^2)\hat{\boldsymbol{\imath}} - 2xy\hat{\boldsymbol{\jmath}}\right]. \tag{54}$$

The force is that on a neutron with its magnetic moment parallel to the line of charge (in the positive z direction) moving in the positive y direction with speed v_o. The electric dipole in this case is given by $\boldsymbol{p} = \mu v_o/c^2\hat{\boldsymbol{\imath}}$.

Boyer considers two such neutrons traveling in the positive y direction with speed v_o, one at $x = +a$ and one at $x = -a$. He assumes the paths will not vary much from straight lines, an assumption that seems justified considering that $\mu_o\mu/2\pi m = 1.15 \times 10^{-6}$ J·m/A·kg, where m is the neutron mass. He finds that a neutron passing on the positive x side of the positively charged wire is delayed with respect to one passing on the negative side by an amount $\Delta y = \mu_o\mu\lambda/mv_o$, which results in the same phase shift as found by Aharonov and Casher in Eq. (51). Hence Boyer claims the phase shift of the AC effect is due to classical lag rather than a quantum topological effect. Aharonov *et al*[22] responded that Boyer overlooked the effect of hidden momentum in the charge-magnet system, which acts to render the net force on the neutron zero.

When Aharonov *et al.* equate the net force acting on the neutron (their equation (6)) to that acting between the line of charge and the induced electric dipole plus that due to the supposed rate of change of the hidden momentum, they find that the net force on the neutron is zero. However, with neither hidden momentum[5,6,23,24] in a charge-magnet system nor an electric dipole on a moving magnetic moment[13,14], the AC effect needs a different explanation.

As shown above, when a point charge is brought into the vicinity of a small Amperian magnet, the magnet gains an amount of mechanical linear momentum $\boldsymbol{\mu} \times \boldsymbol{E}/2c^2$ with an equal momentum gained by the charge, and the opposite of the sum of these is deposited in the electromagnetic field. The force is due to the magnetic field produced by the displacement current as the charge approaches.

When you perform the same steps to get the impulse on the magnet due to the line of charge moving toward the magnet (or vice versa) as was done for the point charge, the momentum transferred to the magnet is found to be

$$\boldsymbol{P}_m = \boldsymbol{\mu} \times \boldsymbol{E}/c^2, \tag{55}$$

where $\boldsymbol{E}$ is the electric field due to the line of charge. As the magnet moves through the field, the mechanical momentum will in general change with time, giving rise to a force on the magnet. With the electric field given by Eq. (53), the force is given by $d\boldsymbol{P}_m/dt$ with x and y components, respectively,

$$F_x = \frac{\mu_o \mu \lambda}{2\pi r^2} \left[\frac{2}{r^2} (\boldsymbol{r} \cdot \boldsymbol{v}) y - \dot{y} \right] = m\ddot{x}, \tag{56}$$

and

$$F_y = \frac{\mu_o \mu \lambda}{2\pi r^2} \left[-\frac{2}{r^2} (\boldsymbol{r} \cdot \boldsymbol{v}) x + \dot{x} \right] = m\ddot{y}, \tag{57}$$

the same as found by Boyer. Here $\boldsymbol{v}$ is the velocity of the magnet and m is the magnet's mass. When you make the same assumptions as Boyer (initial velocity = $\dot{y}\hat{\boldsymbol{j}} = v_o\hat{\boldsymbol{j}}$ and $x = \pm a$), these equations become

$$F_x = \frac{\mu_o \mu \lambda}{2\pi r^4} (y^2 - a^2) = m\ddot{x}, \tag{58}$$

and

$$F_y = \frac{\mu_o \mu \lambda}{2\pi r^4} (\pm a y) = m\ddot{y}. \tag{59}$$

From this point on, if you continue the argument of Boyer, you arrive at his result, that the AC phase shift, Eq. (51), is due to a classical lag.

There are at least two solutions to Eqs. (56) and (57), and one is important for the Aharonov-Casher effect. The trivial solution just has the magnet stationary in the electric field. This would be done my applying mechanical forces to place the magnet at rest in the field.

A more interesting solution can be found by putting Eqs. (56) and (57) in polar coordinates. The equations are then

$$\boldsymbol{F} = -\frac{\mu_o \mu \lambda}{2\pi r^2} (r\dot{\phi}\hat{\boldsymbol{r}} + \dot{r}\hat{\boldsymbol{\phi}}) = m[(\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}}]. \tag{60}$$

Look for a solution where r is constant. The equation reduces to

$$\frac{\mu_o \mu \lambda}{2\pi r} \hat{\boldsymbol{r}} = m r \dot{\phi} \hat{\boldsymbol{r}}. \tag{61}$$

The magnet can therefore execute a circle in the counterclockwise direction (assuming λ is positive) with an angular speed of

$$\omega = \frac{\mu_o \mu \lambda}{2\pi m r^2}. \tag{62}$$

Note that no such orbit exists in the clockwise direction for positive λ.

It is interesting that the circumference of the orbit is exactly the same as the classical lag found by Boyer: $\mu_o \mu \lambda / mv$ where $v = r\omega$. Since this highly improbable orbit is due to electromagnetic-derived forces and not to quantum effects, its theoretical existence supports classical lag as the cause of the Aharonov-Casher effect as claimed by Boyer.

VII. CONCLUDING DISCUSSION

The understanding of momentum in charge-magnet systems has been hampered by not taking the formation of these systems into account. When an Amperian magnet is subjected to an electric field or is formed in a preexisting electric field (or some combination thereof), Lorentz forces arise that will (in general) impart momentum to the magnet and the charges and/or to an external agent exerting mechanical forces on them. An equal and opposite amount of momentum is added to the electromagnetic field. The solution to the Shockley-James paradox[4] is that their charge-magnet system is either not being viewed in its original rest frame or the mechanical momentum that the system would gain is present in an external agent. No hidden momentum resides in the charge-magnet system[5].

It has generally been thought that an Amperian magnet moving in an observer's frame of reference will be observed to have an electric dipole present on it perpendicular to both the magnetic moment and the direction of motion. However, there is no such dipole; its mathematical manifestation is due to ignoring the effects of the relativity of simultaneity[13,14].

The solution to the paradox of Mansuripur[10] is simply that there is no electric dipole on a moving magnetic dipole. Hence the supposed torque that would be seen on a moving charge-magnet system due to the interaction of the charge with the electric dipole is not present. No mechanical torque is seen whether or not the system is in motion[25].

As Furry[3] has shown, a charge-magnet system where the magnet is Amperian can contain both linear and angular momentum in its electromagnetic field. This momentum is obtained when the system is formed and balances the mechanical momentum that is also generated[5]. When the system is in motion there is a torque, but it is in the electromagnetic field and has been mistaken as a mechanical torque acting on the magnet.

There are two equal and opposite sources of torque in a moving charge-magnet system, yielding no net torque. One is due to the motion of the electromagnetic field. The field momentum Furry identified is Lorentz-transformed into a time-dependent angular momentum, thus producing a torque. The other source is due to the interaction between the charge and the current loop of the magnet, not between the charge and an electric dipole on the magnet[25].

A current loop can experience a torque in a magnetic field unless the field and magnetic moment are parallel or antiparallel. A current loop moving through an electric field will be in a Lorentz-transformed magnetic field and thus may experience a torque. An observer at rest with the electric field sees no magnetic field, and it has been thought that the observer would detect no torque, thus creating a paradox. Vaidman[7] claimed to have solved this paradox by appealing to induced mechanical forces.

However, if there actually is torque acting on the current loop in its frame of reference, a Lorentz transformation to any other frame of reference cannot do away with that torque. So, if there is a torque observed in the moving frame due to a magnetic field in that frame, there will also be a torque seen in any other reference frame, whether or not a magnetic field is detected in that frame.

In the Aharonov-Casher effect[20] neutral Amperian magnets (such as exist on neutrons) are supposed to experience a differential phase shift in their wave functions as they pass on either side of a line of charge. According to this effect, the phase shift difference is a result of what is called a quantum topological effect rather than some force that causes one magnet to beat another to a detection system. Boyer[21] disputed the quantum nature of the effect by calculating a lag between neutrons passing either side of the line of charge due to a force between the charge of the line and an electric dipole on the moving magnet. However there is no electric dipole present and this explanation does not work.

Aharonov *et al.*[22] accepted the existence of the force Boyer identified but argued that hidden momentum was involved in canceling it. There is no hidden momentum and no electric dipole, but there is still a force on the moving magnet due to the change in its mechanical momentum to offset the opposite change in the electromagnetic momentum. As such, there is a lag between magnets moving on opposite sides of the line of charge, and this lag turns out to be the same as that calculated by Boyer.

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