Linear and Angular Momentum in Electromagnetic Systems Containing Magnetic Dipoles

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This article is concerned with mechanical and field momentum in systems consisting of magnets and electric fields. Both linear and angular momentum are considered. There have been controversies and contradictions in the physics literature over questions concerning these systems for over 100 years. Many paradoxes have arisen but not all have been solved to everyone's satisfaction. Central to many of these paradoxes are the claims Amperian magnets in electric fields contain hidden mechanical momentum and an electric dipole exists on a moving magnetic dipole. I examine several paradoxes, a controversy surrounding the Aharonov-Casher effect, and consider the nature of the force between magnets.

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I. INTRODUCTION

It has been known for a long time that electromagnetic fields can contain both linear and angular momentum [1, 2]. The linear electromagnetic momentum per unit volume in free space is given in SI units by

$$\wp_{em} = \epsilon_o \boldsymbol{E} \times \boldsymbol{B}, \tag{1}$$

where ϵ_o is the permittivity of free space, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic field (also commonly called the magnetic induction). The quantity of linear electromagnetic field momentum in a region of space requires a volume integral of Eq. (1) over that region. The electromagnetic angular momentum density about a certain point involves taking the vector product between a position vector centered on that point and the linear momentum density at a point in space, that is,

$$\boldsymbol{l}_{em} = \epsilon_o \boldsymbol{r} \times (\boldsymbol{E} \times \boldsymbol{B}). \tag{2}$$

Once again you must integrate over a volume of space to find the electromagnetic angular momentum in that space.

In general the electromagnetic fields will be changing in time and there will be electric and magnetic matter in the integrated volumes. This situation can be handled by utilizing Maxwell's stress tensor, given in SI units by [3].

$$T_{ij} = \epsilon_o \left[\frac{1}{2}\delta_{ij}E^2 - E_iE_j\right] + \frac{1}{\mu_o}\left[\frac{1}{2}\delta_{ij}B^2 - B_iB_j\right], \tag{3}$$

where μ_o is the permeability of free space. (Note: Other authors may define the tensor as the negative of this.) In terms of this tensor, the conservation of linear momentum for a system where there is no mechanical momentum crossing the system boundary can be written as [3],

$$\Sigma_i \boldsymbol{n}_i \left(\Sigma_j \frac{\partial T_{ij}}{\partial x_j}\right) + \boldsymbol{f} + \frac{1}{c^2}\frac{\partial \boldsymbol{S}}{\partial t} = 0. \tag{4}$$

This equation states that the amount of electromagnetic field momentum leaving a unit volume (first term, with $\boldsymbol{n}_i$ as unit vectors) plus the change in the mechanical momentum in that volume (term two) plus the change in electromagnetic field momentum in that volume is zero. Since no particles are allowed into or out of the volume, the increase in the mechanical momentum (term two) is a decrease in the field momentum, which makes $\boldsymbol{f}$ the electromagnetic force density on the particles in the volume. So, if total momentum is to be conserved, an increase in the field momentum in the volume must be exported outside the volume to keep total momentum constant (term one is negative). Alternatively, a decrease in field momentum in the volume must be compensated by field momentum entering from the outside (term one is positive).

The volume integral of the first term of Eq. (4) is the flow of field momentum out of the chosen volume per unit time; the volume integral of $\boldsymbol{f}$ is the transfer of field momentum to mechanical momentum per unit time in the volume; and the integral of the right-most term is the change in field momentum in the volume per unit time. The quantity $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}/\mu_o$ is called the Poynting vector and is the flow of electromagnetic energy per unit area per unit time. For a completely isolated system, the first term in Eq. (4) would be zero; however, due to the long range of the electromagnetic force, this would have to be a whole universe!

The conservation of angular momentum is expressed as the vector product of the position vector about which angular momentum is calculated with Eq. (4). This can be written in a general form as

$$\boldsymbol{r} \times \nabla \cdot \bar{T} + \boldsymbol{r} \times \boldsymbol{f} + \frac{1}{c^2}\boldsymbol{r} \times \frac{\partial \boldsymbol{S}}{\partial t} = 0. \tag{5}$$

Since I am only covering non-radiating systems that are isolated from the rest of the universe, there is no need to worry here about momentum crossing volume surfaces; in other words we do not need Eqs. (4) or (5).

This article is concerned with linear and angular momentum in isolated systems. There are a number of paradoxes associated with these systems that haven't been resolved to everyone's satisfaction. In what follows I will examine momentum in electromagnetic systems containing magnetic dipoles, especially with regard to some of these paradoxes, often in a fully relativistic way.

II. HIDDEN MOMENTUM

It is rather amazing that even though systems consisting of magnetic dipoles in electric fields seem rather simple, paradoxes involving these systems are still being argued about. Problems involving these systems were recognized as long ago as 1891 by J. J. Thompson [1]. Much of the controversy is in how the special theory of relativity is applied to such systems. Electromagnetic theory was the first relativistic theory (in the modern sense), though it was not recognized as such until the work of Einstein [4].

Many of these paradoxes involve the concept of hidden momentum, a term coined by Shockley and James [5] when considering the momentum in a charge-magnet system in 1967. Their model consisted of two equal and opposite charges equally distant on either side of a magnet. The magnet consisted of a couple of oppositely rotating non-conducting disks, one with a positive charge on its rim and the other with an equal negative charge.

According to Eq. (1) there is electromagnetic field linear momentum in this model, but it was just sitting there in their analysis, not moving. So where was the equal and opposite mechanical momentum necessary to balance the electromagnetic momentum? They suggested there must be an unseen relativistic form of momentum in the magnet, the so-called hidden momentum. However, they did not consider how this system came to be in the first place, and that, as I will explain, is where they made an error [6].

Just what is hidden momentum? Babson *et al.* [7] have given a couple of models attempting to explain why hidden momentum is present in a magnet immersed in an electric field. The simpler model is likely the one where there is a rectangular tube with curved edges and area A in which identical charged but non-interacting particles can circulate, creating a current I and a magnetic dipole IA. (See Figure 1, left side.)

In the figure an electric field is directed upward and particles on the left side of the tube are accelerated in that direction. The particles travel across the top of the tube and are decelerated as they move downward toward the bottom of the tube. Hence the particles traversing the top of the tube are moving faster than those traversing the bottom of the tube.

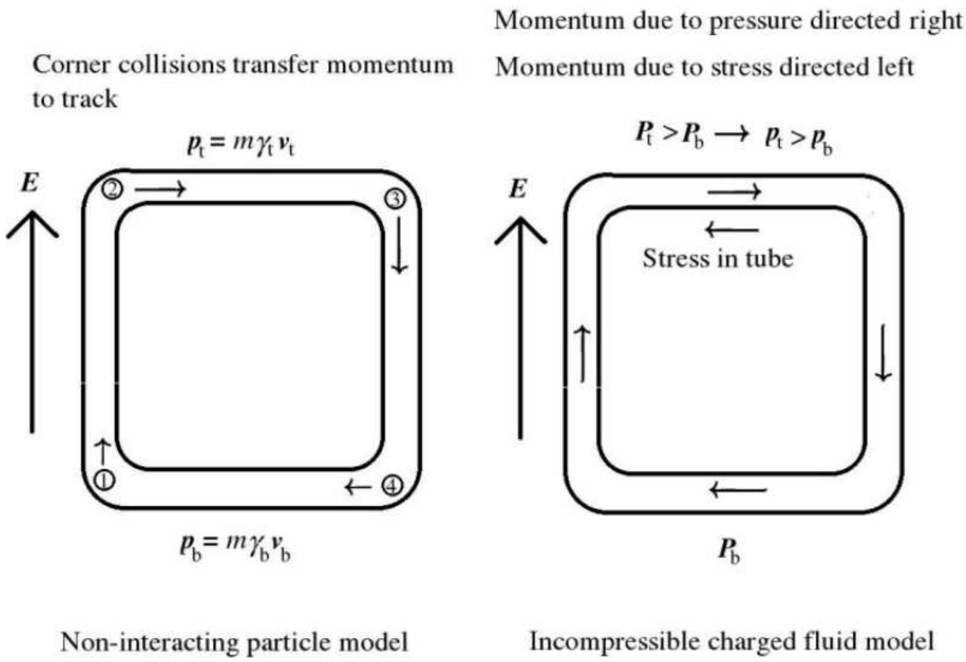

FIG. 1. Models of Hidden Momentum

In order to prevent an unphysical "bunching" of particles, which would mean the current produced by the moving particles would vary from place to place, it is assumed the number of particles passing a given point in the tube per unit time is the same throughout the tube. For each particle in the ascending segment of the tube, there is a particle in the descending segment with equal and opposite momentum. Therefore, there is no net momentum present in the ascending and descending segments.

In the top segment the particles are moving faster than in the bottom, such that each particle at the top has more momentum to the right than a particle at the bottom has to the left. However, the requirement that the current be equal everywhere means the particles at the top are farther apart than those at the bottom.

In Newtonian mechanics momentum is mass times velocity, which means that momentum per unit length at the top to the right is the same as momentum per unit length at the bottom to the left when the current is equal in both segments. However, in relativity theory momentum of a particle is

$$\boldsymbol{p} = m\gamma\boldsymbol{v}, \tag{6}$$

where m is the particle's mass and $\boldsymbol{v}$ is the particle's velocity. The Lorentz factor γ is given by

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{7}$$

so that the momentum at the top is slightly larger in magnitude than at the bottom since γ at the top is slightly larger than γ at the bottom. This is supposed to result in a greater momentum to the right in the top than to the left in the bottom. There is net linear momentum in the magnet that is not seen according to this view. Yet, if the electric field were turned off, the loop would move to the right (if loose) as the hidden momentum was released. In terms of the electric field E, the area of the loop, and the current due to the flow of the non-interacting charges, Babson *et al.* compute a hidden momentum given by

$$p_{hid} = EIA/c^2 \quad \text{or} \quad \boldsymbol{p}_{hid} = \boldsymbol{\mu} \times \boldsymbol{E}/c^2, \tag{8}$$

where $\boldsymbol{\mu} = I\boldsymbol{A}$ is the magnetic moment of the loop and the area vector of the loop, $\boldsymbol{A}$, is defined by the right-hand rule applied to the current I.

This non-interacting particle picture is not directly applicable to a real current loop, so a more realistic model is one where the loop is filled with a charged, incompressible fluid to mimic a real electric current [7]. This model is depicted in Figure 1 on the right. In this model the speed of the fluid has to be the same at every point in the tube (with constant cross section). There is thus no difference in γ at the top and that at the bottom. However the pressure in the fluid increases from bottom to top in the left segment of the tube and decreases from top to bottom in the right segment due to the electric force. Pressure (and stress) can result in momentum in special relativity [8–10].

Babson *et al.* calculate the hidden momentum due to the pressure difference is given by

$$p_{hid} = \gamma^2(P_t - P_b)vla/c^2, \tag{9}$$

where P_t and P_b are the pressures at the top and bottom, respectively, v is the speed of the fluid, l is the length of the top (and bottom) segments of the loop, and a is the cross-sectional area of the tube. Then they show this expression also equals what appears in Eq. (8).

Both models described above suffer from the same error. Note that they are posed as existing systems with no history. What if you assemble these systems from components that have no energy or momentum? Certainly this can be done, and when it is done you get a different picture.

Look at the non-interacting charged particle model. If you apply an electric field to the model where the charges are initially stationary and the tube is nailed down, all you do is push the charges up towards the top. No circular motion will occur. What you can do is introduce the particles one by one inside the tube at the lower left corner (location 1 in the left side of Figure 1). This particle will accelerate upward in the electric field and collide with the upper left corner (2). The particle will bounce to the right with a certain right-directed momentum. The tube will gain an equal and opposite momentum, except, since it is nailed down, that momentum will be transferred to its environment.

When the particle reaches the upper right corner (3) it will deposit its momentum in the tube. Now the left-right momenta of both the particle and the tube are zero as the particle approaches the lower right corner (4). Moving slower due to being decelerated by the electric field, there will be less left-right momentum exchange at this corner, but the momenta will again be equal and opposite: The left-right momentum of the system will continue to be zero. Due to the momentum exchange at the corners of the tube, it is irrelevant whether or not you invoke Newtonian or relativistic physics.

The same sequence of collisions occurs with each particle introduced to the tube at location 1. It turns out as you do this there will be a slightly greater momentum at the top equal to the momentum of one particle there [6]. For a nailed-down tube, the environment will have an equal and opposite momentum; however, this momentum will not be the same as that given in Eq. (8) and will be entirely negligible for a large number of small particles.

I developed the picture above to clarify the momentum interaction at the corners of the particle track. The picture is no different if the particles are already circulating by some means. There will still be momentum exchanges at the corners such that no hidden momentum develops. The only way hidden momentum could be in the model would be if it were prepared specifically for such a situation to hold. In other words the hidden momentum would have to be supplied somehow by an external agent.

The argument against hidden momentum in the incompressible fluid model is similar. First you have to somehow get the fluid circulating in this model, otherwise all that happens is a static pressure regime appearing in the fluid.

When the fluid is caused to circulate, pressure will build up in the top of the tube greater than that in the bottom as the fluid flows, producing a left to right momentum.

However, not only does pressure have momentum in relativity, but so does stress. The pressure buildup will produce a buildup of stress in the tube itself, and this will contain momentum equal and opposite to that in the fluid. This will be true at the top and bottom of the tube. Hence there is no net left/right momentum in the fluid-tube system.

III. MOMENTUM IN A CHARGE-MAGNET SYSTEM

In 1969 Furry [11] calculated the electromagnetic linear and angular field momentum in a charge-magnetic dipole system – for both an Amperian magnet where the magnetism is produced by electric current and a magnetic-pole model where the magnetism is produced by magnetic pole pairs, assuming magnetic monopoles exist. In his calculation a magnetic dipole was situated at the origin of a Cartesian coordinate system with its dipole moment $\boldsymbol{\mu}$ directed at an arbitrary angle in the x-z plane. A point charge q was located at $z = a$. He found the linear field momentum of a charge-Amperian magnet system to be

$$\boldsymbol{p}_{em} = \frac{\mu_o q}{4\pi a^3}\boldsymbol{\mu} \times \boldsymbol{a} = \boldsymbol{E} \times \boldsymbol{\mu}/c^2, \tag{10}$$

in the slow-motion approximation ($\gamma = 1$) where the second expression on the right assumes the electric field is uniform over the extent of the magnetic dipole (small dipole approximation). If the magnet is due to a pair of isolated poles, the linear momentum is zero. Furry found that any distribution of electric field and magnetic-pole pairs would have zero linear field momentum.

The field angular momentum was found to be

$$\boldsymbol{L}_{em} = \frac{\mu_o q}{4\pi}\left[\frac{\boldsymbol{\mu} q}{a} - \frac{(\boldsymbol{\mu}\cdot\boldsymbol{a})\boldsymbol{a}}{a^3}\right] = \frac{\mu_o q \mu sin\theta}{4\pi a}\hat{\boldsymbol{\imath}}, \tag{11}$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{\mu}$ and the right-most expression is due to his choice of coordinates. When the electric field is perpendicular to the magnetic moment, the field angular momentum is

$$\boldsymbol{L}_{em} = \frac{\mu_o q \boldsymbol{\mu}}{4\pi a}. \tag{12}$$

In contrast to the case of linear momentum, the angular field momentum was the same whether the magnetic dipole was Amperian or a magnetic-pole pair.

The presence of linear and angular electromagnetic field momentum in a system consisting of a point charge and a magnetic dipole is hard to understand unless there is some way to balance the field momentum with mechanical momentum. This was what Shockley and James [5] set about to do in their 1967 paper where their model consisted an Amperian magnet made of non-conducting material and two oppositely charged point particles equidistant on either side of the magnet and where the magnetic moment was perpendicular to the line between the two charges.

They noted, implicitly, that the two oppositely charged particles guaranteed that there would be no angular momentum about the point where the magnet was located. With this model they only attempted to propose the presence of (hidden) linear momentum in the magnet. However, it is clear from the models of Babson *et al.* [7] that there is angular momentum in those models if the linear hidden momentum exists. Babson *et al.* neither mention nor explain the presence of this angular momentum, which is also present in the model of Shockley and James.

Again, the problem is ignoring the history of the models – how they were assembled in the first place. I have shown [6] how mechanical momentum is imparted to the model of Shockley and James by bringing the two opposite charges in from a great distance to the vicinity of the magnetic dipole. This momentum is transferred to an external agent if that agent exerts mechanical forces to balance the Lorentz forces acting on the charges and magnet. Here, I will bring a single charge in from a great distance to the vicinity of an Amperian magnet dipole to show that the linear and angular mechanical momentum generated is equal and opposite to that found by Furry [11].

To keep the mathematics simple for the sake of argument, I will have the electric field perpendicular to the magnetic moment. The magnetic dipole will be at the origin of the coordinate system with its moment pointing in the positive z direction. The point charge will be moved slowly (to avoid any radiation from acceleration) along the x axis in the positive direction to a point $x = -r$. The magnetic field at the location of the charge as it moves along the x axis is

$$\boldsymbol{B} = -\frac{\mu_o IA}{4\pi|x|^3}\hat{\boldsymbol{k}}. \tag{13}$$

The Lorentz force on the charge is

$$\boldsymbol{F}_L = qv\hat{\boldsymbol{\imath}} \times \left(-\frac{\mu_o I A}{4\pi|x|^3}\hat{\boldsymbol{k}}\right) = \frac{\mu_o q I A v}{4\pi|x|^3}\hat{\boldsymbol{\jmath}}, \tag{14}$$

where I is the current in the Amperian dipole, A is the area of the current loop, $v = dx/dt$ is the speed of the charge, and the magnetic moment is $\boldsymbol{\mu} = IA\hat{\boldsymbol{k}}$.

The impulse imparted to the charge as it moves along the x axis from $x = -\infty$ to $x = -r$ can be calculated as follows [6].

$$\Delta \boldsymbol{p}_q = \int_{-\infty}^{-r} \boldsymbol{F}_L dt = -\frac{\mu_o q I A}{4\pi}\hat{\boldsymbol{\jmath}} \int_{-\infty}^{-r} \frac{dx}{x^3} = \boldsymbol{\mu} \times \boldsymbol{E}/2c^2, \tag{15}$$

where $\boldsymbol{E}$ is the electric field at the location of the dipole, which is considered to be sufficiently small so the field is uniform across the current loop.

There has been some controversy over the correct formula for calculating the force on a magnetic dipole [12–14], so I will calculate it from a more fundamental perspective. The result is the same as that found from the traditional formula, given by

$$\boldsymbol{F} = \nabla(\boldsymbol{\mu} \cdot \boldsymbol{B}), \tag{16}$$

the formula supported by Franklin [12]. The moving charge q produces a magnetic field at the dipole. This is the magnetic field that would be used in the above equation to calculate the force.

The force can be calculated from the magnetic field produced by the displacement current acting on the charge current in the current loop of the magnet (Figure 2). In the slow-motion approximation where $\gamma = 1$, the displacement due to q at the magnetic dipole is

$$\boldsymbol{D} = \frac{1}{4\pi}\frac{q}{r^2}\hat{\boldsymbol{\imath}}, \tag{17}$$

where $r = |x|$ is the distance between the charge and the dipole. The displacement current density is equal to the time rate of change of the displacement, $\boldsymbol{J}_D = d\boldsymbol{D}/dt$. Integrating the inner product of the displacement current density with an area gives the displacement current through that area. Then Ampere's law is used to calculate the magnetic field around the area.

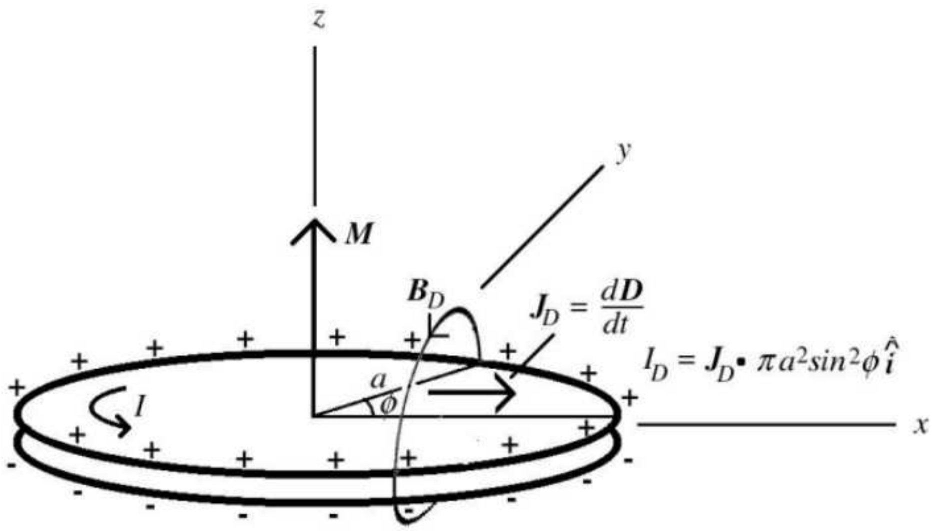

FIG. 2. Displacement Current and Magnetic Field at Magnet Model

In this case you have circular magnetic field lines centered on the x axis and oriented in a counterclockwise direction with respect to the positive x direction. The strength of the field of a magnetic loop will depend on the displacement current within it. The magnetic field loops acting on the charge current will have cross-sectional areas defined by circles with a diameter equal to the distance between the edges of the disks parallel to y (Figure 2). The cross-sectional area of a given loop is $Asin^2\phi = \pi a^2 sin^2\phi$, where a is the radius of the disk and ϕ is the usual azimuth angle of a spherical coordinate system. The displacement current as a function of ϕ involved in the interaction is therefore

$$I_D = \frac{d\boldsymbol{D}}{dt} \cdot \hat{\boldsymbol{\imath}}\pi a^2 sin^2\phi = \frac{qva^2}{2r^3} sin^2\phi, \tag{18}$$

where $\boldsymbol{v} = (-dr/dt)\hat{\boldsymbol{\imath}}$ when you take the time derivative of $\boldsymbol{D}$. From the integral form of Ampere's law, you find the magnetic field at the rims of the disks as a function of ϕ,

$$\boldsymbol{B}_D = \frac{\mu_o I_D}{4\pi a sin\phi}\hat{\boldsymbol{k}} = \frac{\mu_o qav}{4\pi r^3} sin\phi\hat{\boldsymbol{k}}. \tag{19}$$

Now you integrate the Lorentz force around the current loop to get the total force due to the magnetic field of the displacement current.

$$\boldsymbol{F}_D = I\oint d\boldsymbol{l} \times \boldsymbol{B}_D = I\oint ad\phi\hat{\boldsymbol{\phi}} \times \boldsymbol{B}_D = \frac{\mu_o qIAv}{4\pi r^3}\hat{\boldsymbol{\jmath}}, \tag{20}$$

where $\hat{\boldsymbol{\phi}} = -sin\phi\hat{\boldsymbol{\imath}} + cos\phi\hat{\boldsymbol{\jmath}}$. To get the impulse on the dipole, you integrate over time.

$$\Delta\boldsymbol{p}_\mu = \frac{\mu_o qIA}{8\pi r^2}\hat{\boldsymbol{\jmath}} = \boldsymbol{\mu} \times \boldsymbol{E}/2c^2. \tag{21}$$

Note that this is equal to the impulse applied to the charge, Eq. (15), both in magnitude and direction, and when the impulses are added together, you get a result that is equal and opposite the field electromagnetic momentum of Furry, Eq. (10). So, you have started out with components with zero momentum and end up with a charge magnet system with zero momentum, as the field and mechanical contributions to the momentum cancel. (The mechanical forces used to assemble the system are equal and opposite and don't impart momentum.)

There are a variety of ways to model an Amperian magnetic dipole [13], but note that the momentum imparted to the magnet by the above mechanism is not dependent on the model. Even if the current is inside a conducting ring such that the electric field of the charge does not penetrate the current, it is only the electric field external to the magnet that is involved in producing the magnetic field acting on the current.

The next task is to calculate the mechanical angular momentum the charge-dipole system acquires as the point charge is brought in from a great distance. In this case you must also have an external agent to assemble the system, but the agent must acquire the mechanical angular momentum itself so that the final configuration of the charge-dipole system is stationary as it is in Furry's calculation. (Actually, you would need to use the external agent in the previous calculation to ensure that the charge-magnet system ended up in Furry's configuration unless the masses of the charge and magnet were equal. Then the relative configuration would be correct but the rest frame would have changed.)

As the point charge q is moved in toward the magnetic dipole, it experiences the Lorentz force given by Eq. (14). The dipole will experience an equal force. Both forces will be countered by the external agent, such that the magnetic dipole remains stationary and the point charge moves in a straight line along the x axis in the positive x direction. As in the calculation of Furry, the angular momentum will be taken about the location of the magnetic dipole. The force of the external agent, of course, produces no angular momentum about the location of the dipole. The mechanical angular momentum is given by

$$\boldsymbol{L}_m = \int \boldsymbol{\tau}_m dt = \int \boldsymbol{r} \times \boldsymbol{F}_L dt = -\frac{\mu_o q\mu\hat{\boldsymbol{k}}}{4\pi}\int_{-\infty}^{-r}\frac{dx}{x^2} = -\frac{\mu_o q\boldsymbol{\mu}}{4\pi r}. \tag{22}$$

This mechanical angular momentum is equal and opposite to the field angular momentum found by Furry Eq. (12). No hidden momentum, either linear or angular, is necessary for momentum conservation.

IV. THE MANSURIPUR PARADOX AND CHARGE SEPARATION ON MOVING MAGNETIC DIPOLES

A news article that appeared in the 27 April 2012 issue of the journal *Science* [15] reviewed a claim that provoked a lot of discussion among many researchers involved in electromagnetic theory and special relativity. A researcher at the University of Arizona, Masud Mansuripur, claimed the Lorentz force of electromagnetism was not compatible with special relativity [16]. His argument was based on a paradox involving angular momentum in a charge-magnet system.

The paradox is depicted in Figure 3. In the inertial reference frame S', observer O' is stationary with respect to a charge q at the origin and a magnetic dipole $\boldsymbol{\mu}$ at $x' = a$. She sees no reason for there to be any interaction between the charge and the dipole. However, frame S' is moving to the right in the inertial frame of observer O, and, according to the conventional idea, he should see an electric dipole on the magnetic dipole, indicated by the charge symbols in Figure 3. In his frame of reference, he should see the positive side of the magnetic dipole repelled by charge q and the negative side attracted resulting in a torque acting on the magnetic dipole – a torque which is not observed by O'. This situation is clearly outlawed by the principle of relativity, so Mansuripur claims the Lorentz force responsible for the torque is not in tune with relativistic principles and should be replaced by another force, the one proposed by Einstein and Laub [17].

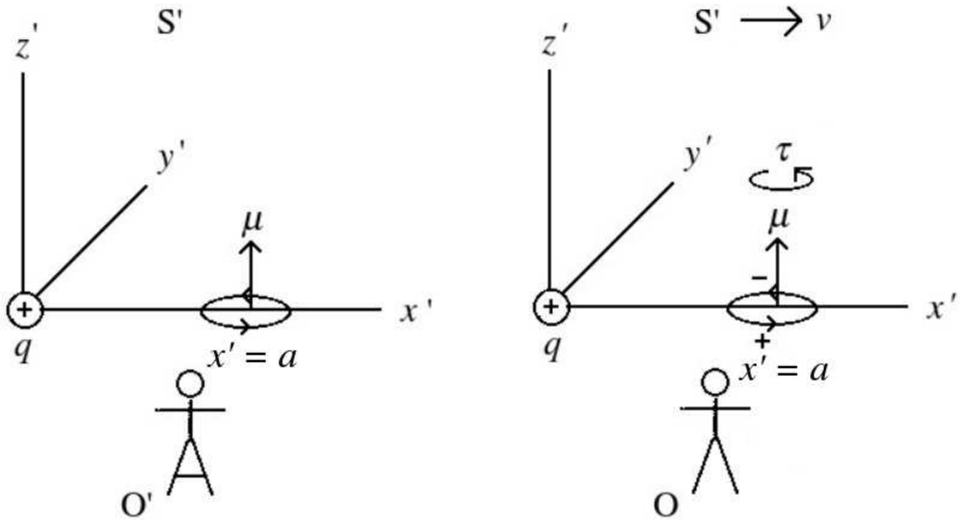

FIG. 3. The Mansuripur Paradox

A resolution to the paradox based on hidden momentum has been proposed. (See, for example, Griffiths and Hnizdo [18].) However, the simplest resolution involves the proposal that an electric dipole is not present on a moving magnetic dipole [19, 20]. With no electric charge separation, there is no torque in either the S or S' frames.

It is interesting that the belief there is an electric dipole on a moving magnetic dipole arises from a misapplication of relativity. When current density is Lorentz-transformed from an inertial reference frame in which there is no charge density to a frame moving with respect to that frame, the general result appears to be a charge density in the moving frame. The charge density four-vector in the S' frame is

$$j^{\mu'} = (j^{1'}, j^{2'}, j^{3'}, c\rho'), \tag{23}$$

where $j^{i'}$ is the current density in the x', y', and z' directions and ρ' is the charge density (in the time component of the four-vector). A Lorentz transformation to the S frame in which the S' frame is moving in the x direction with speed v is [21]

$$j^{\mu} = \gamma(j^{1'} - v\rho', j^{2'}, j^{3'}, c\rho' - vj^{1'}/c). \tag{24}$$

According to the above equation, there is a charge density in the S frame given by $-\gamma v j^{1'}/c^2$ even when ρ' is zero. This implies that the observer O sees the near side of the current loop in Figure 3 to be positive and the far side to be negative, implying he should see the result of a torque acting on the loop due to interaction with charge q.

This point of view is strengthened by considering the Lorentz transformation of the magnetization-polarization four-tensor [21]. This tensor is assumed to be fully relativistic and therefore Lorentz-transformable from one frame of reference to another. When this is done, a substance with only magnetization in its rest frame appears to generally have an electric polarization in a frame in relative motion. This is supposed to mean a magnetic dipole in motion should be seen to have an electric dipole with a dipole moment perpendicular to both the magnetic dipole moment and the direction of relative motion.

Franklin [19] has produced a direct explanation as to why this is not true. He shows that Eq. (24) is not correct as naively interpreted because the current density is a function of the space and time coordinates which have to be transformed along with the current density itself. By not performing this transformation, the relativity of simultaneity is violated.

To see this is the case, consider Figure 4, which depicts a rectangular current loop in its rest frame, S'. The left-right length of the loop is l' and it carries a counterclockwise positive current I'. In the center of the bottom segment of the loop is a pulse generator that emits brief light pulses to the left and right simultaneously.

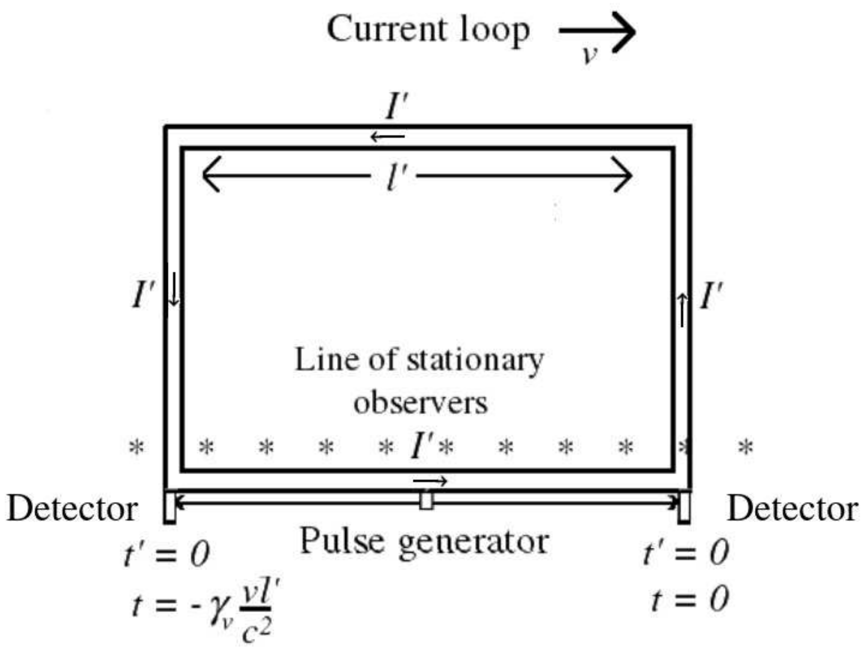

FIG. 4. A Moving Current Loop

In the S' frame the pulses arrive at the same time at the detectors, but that is not the case in the lab (S) frame in which the loop is moving to the right with speed v. In the S frame, due to the fact the speed of light is the same in all frames, the left-moving pulse arrives at the detector at the trailing end of the loop at a time $\gamma v l'/c^2$ before the right-moving pulse arrives at the detector at the leading end of the loop. This is nothing more than the proverbial "train paradox" of relativity.

It can be shown [20] that an amount of charge $(vl'/c^2)I'$ passes the left side of the loop after the pulse arrives there and before the pulse arrives at the right side of the loop. A similar situation with an opposite amount of charge occurs on the top segment of the loop, making the loop appear to have a charge separation in the S frame. The amount of charge on the top and bottom loop segments determined between the times of the pulse arrivals is different in the S frame from that in the S' frame. The apparent charge separation in the S frame is just the amount necessary for an electric dipole to appear on the loop [20]. However, it is clear this charge is only due to the relativity of simultaneity – due to the fact that the pulse arrivals are not simultaneous in the S frame as they are in the S' frame.

I have also questioned the relativistic nature of the magnetization-polarization tensor [20]. If the magnetization of a substance is Amperian – that is, due to current rather than magnetic charges – the argument of Franklin [19] and that given just above would apply: The Lorentz-transformation of the tensor is not properly done without a concomitant transformation of the coordinates of the tensor components. As such, the magnetization-polarization tensor is relativistically covariant if (and only if) a Lorentz transformation of the coordinates accompanies the Lorentz transformation of the tensor components. The same is, of course, true of the current density four-vector, Eq. (23).

V. THE ELECTRIC AND MAGNETIC FIELDS OF A MOVING MAGNETIC DIPOLE

It can be shown that the fields of a moving magnetic dipole are not compatible with the idea that an electric dipole exists on the magnetic dipole. The magnetic field of a point magnetic dipole located at the origin of a coordinate system in its rest frame (S') is

$$\boldsymbol{B}' = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{\mu}'\cdot\boldsymbol{r}')\boldsymbol{r}'}{r'^5} - \frac{\boldsymbol{\mu}'}{r'^3}\right] - \frac{2\mu_o\boldsymbol{\mu}'}{3}\delta(\boldsymbol{r}'), \tag{25}$$

where $\boldsymbol{\mu}' = \mu'\hat{\boldsymbol{k}}$, $\boldsymbol{r}' = x'\hat{\boldsymbol{\imath}} + y'\hat{\boldsymbol{\jmath}} + z'\hat{\boldsymbol{k}}$, and $\delta(\boldsymbol{r}')$ is the Dirac delta function, which accounts for the singularity at the location of the dipole. Since this term has no effect on the fields away from the dipole, it can be ignored. If a current loop forming a magnetic dipole is sufficiently small, the above equation can be considered to be the field of the current loop. Let the current loop be moving in the positive x direction in the reference frame S (lab frame) such that it is at the origin of both coordinate systems at time $t' = t = 0$. The magnetic field components in the lab frame are given by

$$\boldsymbol{B} = B_{x'}\hat{\boldsymbol{\imath}} + \gamma B_{y'}\hat{\boldsymbol{\jmath}} + \gamma B_{z'}\hat{\boldsymbol{k}} \tag{26}$$

and

$$\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} = \gamma v B_{z'}\hat{\boldsymbol{\jmath}} - \gamma v B_{y'}\hat{\boldsymbol{k}}. \tag{27}$$

The S' position vector transforms to $\boldsymbol{r}' = \gamma x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ so that

$$\begin{aligned} r'^2 &= \gamma^2 r^2 - (\gamma^2 - 1)(y^2 + z^2) = \gamma^2 r^2[1 - (v^2/c^2)\sin^2\alpha] \\ &= r^2[1 - \gamma^2(v^2/c^2)cos^2\alpha], \end{aligned} \tag{28}$$

where $\boldsymbol{r} = x\hat{\boldsymbol{k}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ is the position vector in the lab frame and α is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$ [20].

Taking $\boldsymbol{\mu}'$ to be in the positive z direction, the components of the magnetic field in the primed frame are

$$\begin{aligned} B_{x'} &= \frac{3\mu_o m'}{4\pi}\frac{x'z'}{r'^5}, \\ B_{y'} &= \frac{3\mu_o m'}{4\pi}\frac{y'z'}{r'^5}, \\ B_{z'} &= \frac{\mu_o m'}{4\pi}\left[\frac{3z'^2}{r'^5} - \frac{1}{r'^3}\right]. \end{aligned} \tag{29}$$

The transformed field components are,

$$\begin{aligned} B_x &= \frac{\mu_o\mu}{4\pi}\frac{3xz}{r^5[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{5/2}}, \\ B_y &= \frac{\mu_o\mu}{4\pi}\frac{3yz}{r^5[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{5/2}}, \\ B_z &= \frac{\mu_o\mu}{4\pi}\left[\frac{3z^2}{r^5[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{5/2}} - \frac{1}{r^3[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{3/2}}\right]. \end{aligned} \tag{30}$$

and

$$\begin{aligned} E_x &= 0, \\ E_y &= \frac{\mu_o v\mu}{4\pi}\left[\frac{3z^2}{r^5[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{5/2}} - \frac{1}{r^3[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{3/2}}\right]. \\ E_z &= -\frac{\mu_o v\mu}{4\pi}\frac{3yz}{r^5[1 - \gamma^2(v^2/c^2)cos^2\alpha]^{5/2}}, \end{aligned} \tag{31}$$

Noting that $(\hat{\boldsymbol{k}} \cdot \boldsymbol{r})\boldsymbol{r} = xz\hat{\boldsymbol{\imath}} + yz\hat{\boldsymbol{\jmath}} + z^2\hat{\boldsymbol{k}}$, allows you to the express the equations in (30) in a coordinate-free way as

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\gamma\boldsymbol{\mu}' \cdot \boldsymbol{r})\boldsymbol{r}}{r^5[1 + \gamma^2(v^2/c^2)\cos^2\alpha]^{5/2}} - \frac{\gamma\boldsymbol{\mu}'}{r^3[1 + \gamma^2(v^2/c^2)\cos^2\alpha]^{3/2}}\right]. \tag{32}$$

In the slow-motion approximation you can set the quantities in the brackets in the denominators equal to one, such that the transformed dipole is $\boldsymbol{\mu} = \gamma\boldsymbol{\mu}'$. However, little appears to be gained by this since γ is taken to be one in the slow-motion approximation anyway.

The electric field resulting from the transformation, $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$ at the moment the axes of the laboratory and primed frames are aligned is

$$\boldsymbol{E} = \frac{\mu_o}{4\pi}\left[-\frac{3\gamma(\boldsymbol{\mu}' \cdot \boldsymbol{r})\boldsymbol{v} \times \boldsymbol{r}}{r^5[1 + \gamma^2(v^2/c^2)\cos^2\alpha]^{5/2}} + \frac{\gamma\boldsymbol{v} \times \boldsymbol{\mu}'}{r^3[1 + \gamma^2(v^2/c^2)\cos^2\alpha]^{3/2}}\right]. \tag{33}$$

The usual definition of the electric dipole resulting from the motion of a magnetic dipole is $\boldsymbol{p} = \gamma\boldsymbol{v} \times \boldsymbol{\mu}'/c^2$ such that there should be a term $3\gamma(\boldsymbol{v} \times \boldsymbol{\mu}' \cdot \boldsymbol{r})\boldsymbol{r}/r^5$ inside the brackets if Eq. (33) is the equation of an electric dipole field. To introduce $\boldsymbol{p}$, the numerator of the first term in brackets can be written using a vector identity as $(\boldsymbol{\mu}'\cdot\boldsymbol{r})(\boldsymbol{v}\times\boldsymbol{r}) = -(\boldsymbol{v}\times\boldsymbol{\mu}'\cdot\boldsymbol{r})\boldsymbol{r}+(\boldsymbol{v}\times\boldsymbol{\mu}')r^2-(\boldsymbol{r}\times\boldsymbol{\mu}')(\boldsymbol{v}\cdot\boldsymbol{r})$. Substituting this and the expression for $\boldsymbol{p}$ in the above equation results in

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}-3\boldsymbol{p}r^2+3(\boldsymbol{r}\times\boldsymbol{\mu}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5[1+\gamma^2(v^2/c^2)\cos^2\alpha]^{5/2}}+\frac{\boldsymbol{p}}{r^3[1+\gamma^2(v^2/c^2)\cos^2\alpha]^{3/2}}\right]. \tag{34}$$

For the slow-motion case, this equation can be expressed as

$$\boldsymbol{E} = \frac{1}{4\pi\epsilon_o}\left[\frac{3(\boldsymbol{p}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}-\frac{\boldsymbol{p}}{r^3}+\frac{3(\boldsymbol{r}\times\boldsymbol{\mu}')(\boldsymbol{v}\cdot\boldsymbol{r}/c^2)}{r^5}-\frac{\boldsymbol{p}}{r^3}\right]. \tag{35}$$

The first two terms in brackets are the terms of a point electric dipole, and if those were the only terms, you could argue there is an electric dipole present on a moving magnetic dipole, at least when the speed of the dipole was much less than c. However the last two terms, though dipole-like, spoil the dipole field. In particular, there is no electric field parallel to the $\boldsymbol{v}$ direction.

Applying the same procedure to an electric dipole field will obviously render a similar result. The transformed (slow-motion approximation) magnetic field is

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{3(\boldsymbol{\mu}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5}-\frac{\boldsymbol{\mu}}{r^3}+\frac{3(\boldsymbol{r}\times\boldsymbol{p}')(\boldsymbol{v}\cdot\boldsymbol{r})}{r^5}-\frac{\boldsymbol{\mu}}{r^3}\right]. \tag{36}$$

The same result is obtained if you create an electric dipole out of two equal and opposite charges, letting the charge magnitude go to infinity as the separation of the charges goes to zero while holding their product constant, instead of starting out with the point electric dipole equation. Hence there is neither an electric dipole on a moving magnetic dipole nor a magnetic dipole on a moving electric dipole. Rather, the fields due to the moving dipoles are found from Maxwell's equations, which, of course, has to be the case.

VI. TORQUE IN A MOVING CHARGE-MAGNETIC DIPOLE SYSTEM

One area of possible confusion over the existence or non-existence of torque in a moving charge-magnetic dipole system, such as that which arises in the Mansuripur paradox, could be the fact that there is indeed torque in the system. The system contains linear and angular momentum in its electromagnetic field and both appear in the angular momentum four-tensor. When the system is moving, this angular momentum is changing, giving rise to torque. Also, there is indeed torque due to the interaction between the charge and the magnetic dipole when the system is moving, but this torque is also in the electromagnetic field and is not mechanical torque. These two torques are equal and opposite, canceling out with the result the angular momentum of the system is conserved with no mechanical torque present.

First, look at the field angular momentum. The angular momentum four-tensor is given by

$$L^{\mu\nu} = \begin{pmatrix} 0 & L_z & -L_y & mcx-ctp_x \\ -L_z & 0 & L_x & mcy-ctp_y \\ L_y & -L_x & 0 & mcz-ctp_z \\ mcx+ctp_x & mcy+ctp_y & mcz+ctp_z & 0 \end{pmatrix}. \tag{37}$$

Here, $\boldsymbol{L}$ is the angular momentum of the system, $\boldsymbol{p}$ its linear momentum, m is the system mass, (x, y, z) is the point about which the angular momentum is taken with respect to the center of mass, and t is the time in the rest frame of the system. Consider the charge-magnet system of the Mansuripur paradox. The angular momentum four-tensor for the frame S' is, in the slow-motion approximation,

$$L^{\mu'\nu'} = \begin{pmatrix} 0 & \frac{\mu_o q\mu}{4\pi a} & 0 & mca \\ -\frac{\mu_o q\mu}{4\pi a} & 0 & 0 & \frac{\mu_o q\mu ct}{4\pi a^2} \\ 0 & 0 & 0 & 0 \\ -mca & -\frac{\mu_o q\mu ct}{4\pi a^2} & 0 & 0 \end{pmatrix}, \tag{38}$$

where Eqs. (10) and (11) have been used. Transforming this tensor to the lab frame S in which frame S' is moving with speed v in the positive x direction, you get

$$L^{\mu\nu} = \begin{pmatrix} 0 & \frac{\mu_o q\mu}{4\pi a} - \frac{\mu_o q\mu vt}{4\pi a^2} & 0 & mca \\ -\frac{\mu_o q\mu}{4\pi a} + \frac{\mu_o q\mu vt}{4\pi a^2} & 0 & 0 & \frac{\mu_o q\mu ct}{4\pi a^2} + \frac{\mu_o qv\mu}{4\pi ca} \\ 0 & 0 & 0 & 0 \\ -mca & -\frac{\mu_o q\mu ct}{4\pi a^2} - \frac{\mu_o qv\mu}{4\pi ca} & 0 & 0 \end{pmatrix}, \tag{39}$$

The z component of the angular momentum in the lab frame, $L_z = L^{12}$ is

$$L_z = \frac{\mu_o q\mu}{4\pi a} - \frac{\mu_o q\mu vt}{4\pi a^2}. \tag{40}$$

This angular momentum must be in the electromagnetic field since the first term on the right certainly is and the second is found from a Lorentz transformation of a linear momentum in the electromagnetic field. The time derivative of this gives the time rate of change of the angular momentum, which is the torque involved.

$$\frac{dL_z}{dt} = -\frac{\mu_o vq\mu}{4\pi a^2}. \tag{41}$$

As mentioned, there is torque in the interaction between the current loop and the point charge q, although this torque, like that given in Eq. (41), is in the electromagnetic field and cancels that of Eq. (41). Look at the force four-vector on the current loop in the S' frame. Use the slow-motion approximation such that the current loop is centered on $x = x' = a$ at $t = t' = 0$ in both frames and where quantities unchanged between S and S' in this approximation are not primed. If the radius of the current loop is R, the electric field at a point on the loop $x = a + Rcos\phi$ and $y = Rsin\phi$ due to the charge q, where ϕ is the local azimuth angle measured in the positive direction from the x axis, is given by

$$\boldsymbol{E}' = \frac{1}{4\pi\epsilon_o}\frac{q(\boldsymbol{a}+\boldsymbol{R})}{(a^2+R^2+2aRcos\phi)^{3/2}}, \tag{42}$$

where $\boldsymbol{a} = a\hat{\boldsymbol{i}}$ and $\boldsymbol{R} = R(cos\phi\hat{\boldsymbol{i}} + sin\phi\hat{\boldsymbol{j}})$. The loop carries a current density given by

$$J^{\mu'} = \rho u'(-sin\phi, cos\phi, 0, 0), \quad \text{that is,} \quad J_{x'} = -\rho u' sin\phi \quad \text{and} \quad J_{y'} = \rho u' cos\phi, \tag{43}$$

where ρ is the charge density of the current and u' is the drift speed. Breaking up the electric field into x and y components (no z component is present at the loop) and applying the Lorentz electromagnetic field tensor, you get

$$E^{\mu'\nu'}J_{\nu'} = \begin{pmatrix} 0 & 0 & 0 & \frac{E_{x'}}{c} \\ 0 & 0 & 0 & \frac{E_{y'}}{c} \\ 0 & 0 & 0 & 0 \\ -\frac{E_{x'}}{c} & -\frac{E_{y'}}{c} & 0 & 0 \end{pmatrix}\begin{pmatrix} -J_{x'} \\ -J_{y'} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{J_{x'}E_{x'}}{c} + \frac{J_{y'}E_{y'}}{c} \end{pmatrix} \tag{44}$$

The force density in the time slot is seen to be

$$f_{ct'} = \frac{J_{x'}E_{x'}}{c} + \frac{J_{y'}E_{y'}}{c}. \tag{45}$$

Assuming the distance a is much greater than the loop radius R, the electric field components on the loop in S' are approximately (Eq. (42))

$$E_{x'} \approx \frac{q(a + Rcos\phi)}{4\pi\epsilon_o a^3} \quad \text{and} \quad E_{y'} \approx \frac{qRsin\phi}{4\pi\epsilon_o a^3}. \tag{46}$$

When you substitute $E_{x'}$ and $E_{y'}$ from the above equations and $J_{x'}$ and $J_{y'}$ from Eq. (43) into Eq. (45) and integrate over the volume, you find that the total four-force on the loop in S' is zero due to the angular dependence on ϕ. A four-vector that is zero in one frame of reference has to be zero in all other inertial reference frames, including, of course, the lab frame.

Nevertheless this force is responsible for the appearance of a torque in the lab frame, but this torque results from a force density in the time component of the four-vector rather than a space component, which implies it is not a mechanical torque but one confined to the electromagnetic field. The components of the antisymmetric torque four-tensor, given by the volume integral

$$\tau^{\alpha\beta} = \int_V (x^\alpha f^\beta - x^\beta f^\alpha) dV, \tag{47}$$

in S' acting on the current loop are not all zero. The volume integrals of the torque density that are zero are due to the ϕ dependence and the fact that $z = 0$. The non-zero pair (symmetric-antisymmetric partners) are $\tau^{2'4'}$ and $\tau^{4'2'} = -\tau^{2'4'}$). The calculation of $\tau^{2'4'}$ is carried out as follows, taking the origin about the center of the loop for the volume integration of the torque density,

$$\tau^{2'4'} = \int_{V'} (y' f_{ct'} - ct' f'_y) dV' = \int_{V'} y' f_{ct'} dV' = \int_{V'} (Rsin\phi) \left(\frac{J_{x'} E_{x'}}{c} + \frac{J_{y'} E_{y'}}{c} \right) dV'. \tag{48}$$

To perform the volume integration, you assume that the wire of the loop is one-dimensional, which lets you make the substitution $\rho dV' = \lambda R d\phi$ where λ is the linear charge density of the charge carriers responsible for the current. This allows you to write the integral as

$$\tau^{2'4'} = \frac{R^2 \lambda u}{c} \int_0^{2\pi} (-E_{x'} sin^2\phi + E_{y'} sin\phi cos\phi) d\phi. \tag{49}$$

The second integrand gives zero when integrated over ϕ. The first integrand gives

$$\tau^{2'4'} = \frac{R^2 \lambda u'}{c} \int_0^{2\pi} \left(-\frac{q(a + Rcos\phi)}{4\pi\epsilon_o a^3} \right) sin^2\phi d\phi = -\frac{q\pi\mu/c}{4\pi\epsilon_o a^2}, \tag{50}$$

where $\mu = I\pi R^2 = \lambda u' \pi R^2$. This torque, when transformed to the S frame, gives rise to a torque about the z axis, as follows,

$$\tau_z = \tau^{12} = \frac{v}{c} \tau^{4'2'} = \frac{v}{c} (-\tau^{2'4'}) = \frac{\mu_o v q \mu}{4\pi a^2}. \tag{51}$$

This is the torque that is supposed to be mechanical in nature and produced by the interaction between the charge q and the electric dipole on the moving magnetic dipole. However, this torque is actually in the electromagnetic field, not mechanical, and offsets the torque given in Eq. (41).

VII. A CURRENT LOOP MOVING IN A UNIFORM ELECTRIC FIELD

In this paradox you have a uniform electric field ($\boldsymbol{E} = E\hat{\boldsymbol{k}}$) directed parallel to the positive z axis and a current loop initially moving in the positive x direction in the lab frame (S) at speed v with its magnetic dipole $\boldsymbol{\mu} = \mu\hat{\boldsymbol{\imath}}$ pointed in the direction of motion. This paradox has been treated by Bedford and Krumm [22], by Namias [23]), by Vaidman [13], and by Franklin [19]. In the rest frame of the loop the electric field is moving in the negative x direction with speed v. Hence there is a Lorentz-transformed magnetic field present at the loop (Figure 5).

$$\boldsymbol{B}' = -\gamma \boldsymbol{v} \times \boldsymbol{E}/c^2 = \gamma(v/c^2) E\hat{\boldsymbol{\jmath}} \tag{52}$$

There is also a transformed electric field given by

$$\boldsymbol{E}' = \gamma \boldsymbol{E} = \gamma E\hat{\boldsymbol{k}}, \tag{53}$$

The presence of the magnetic field in the S' frame implies there is a torque on the current loop in that frame (so long as the induced electric field is not screened) given by

$$\boldsymbol{\tau}' = \boldsymbol{\mu}' \times \boldsymbol{B}' = \mu \frac{v}{/} c^2 E\hat{\boldsymbol{k}}, \tag{54}$$

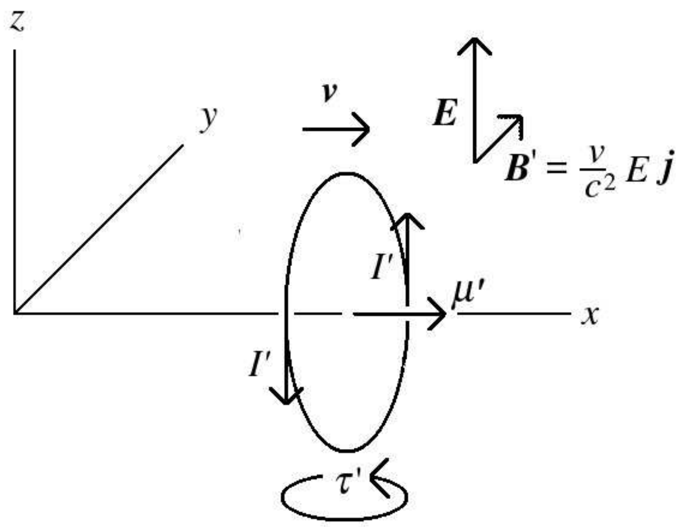

FIG. 5. The "Paradox" Treated by Vaidman

where the term on the far right uses the slow-motion approximation; that is, $\gamma = 1$ ($v << c$), meaning $\boldsymbol{\mu}' = \boldsymbol{\mu}$ and $\boldsymbol{E}' = \boldsymbol{E}$. This approximation will be employed for the rest of this section.

The problem is there is no magnetic field in the lab frame and therefore (presumably) no torque. Why is it that an observer in S' records a torque that is not observed in the lab frame? Vaidman found a resolution for three versions of a magnetic dipole; however, he also claims that two of the dipole models contain hidden momentum. It is possible to resolve the paradox in a general say and show that hidden momentum, if it exists, spoils the resolution.

The angular momentum four-tensor given by Eq. (37) is repeated here for convenience.

$$L^{\mu\nu} = \begin{pmatrix} 0 & L_z & -L_y & mcx - ctp_x \\ -L_z & 0 & L_x & mcy - ctp_y \\ L_y & -L_x & 0 & mcz - ctp_z \\ mcx + ctp_x & mcy + ctp_y & mcz + ctp_z & 0 \end{pmatrix}. \tag{55}$$

Here, as before, $\boldsymbol{L}$ is the angular momentum of the system, $\boldsymbol{p}$ its linear momentum, m is the system mass, (x, y, z) is the point about which the angular momentum is taken measured from the center of mass, and t is the time in the rest frame of the system. The hidden linear momentum in the loop in the S' frame is given by Vaidman as [13]

$$\boldsymbol{P}_{hidden} = -\frac{1}{c}\int \phi \boldsymbol{J} dV = \boldsymbol{\mu} \times \boldsymbol{E}/c^2 = -\mu E/c^2 \hat{\boldsymbol{j}}, \tag{56}$$

when the magnetic dipole is parallel to the positive x direction. $\boldsymbol{J}$ is the current density in the loop and $\phi = -zE$ is the electric potential. (According to the above equation, the hidden momentum will change direction as $\boldsymbol{\mu}$ rotates.) If there is a time-dependent angular momentum $L_{z'}$ along the z axis about the center of the loop at time t ($= t'$ for the slow-motion approximation) and also hidden linear momentum in the loop given at that instant by Eq. (56), the angular momentum four-tensor in S' is

$$dL^{\mu\nu} = \begin{pmatrix} 0 & L_{z'} & 0 & 0 \\ L_{z'} & 0 & 0 & \mu Et/c \\ 0 & 0 & 0 & 0 \\ 0 & -\mu Et/c & 0 & 0 \\ . \end{pmatrix} \tag{57}$$

The torque on the loop is the time rate of change of its angular momentum in the rest frame (S') of the loop (dL_z/dt). When the angular momentum four-tensor is Lorentz-transformed to S frame, the four-tensor will contain

$$L_z = L_{z'} - (v/c^2)\mu E t \tag{58}$$

in the x-y slot. The time derivative of this equation gives

$$dL_z/dt = dL_{z'}/dt - (v/c^2)\mu E, \tag{59}$$

such that the torque in the S frame does not equal that in th S' frame. So, we are back to the paradox of observers in different frames measuring different torques.

The problem is no hidden momentum exists in the current loop in S'. I have shown [6] that you cannot apply an electric field to a magnet without imparting mechanical linear momentum to it unless the magnet is held stationary by an external agent. In that case the external agent is the recipient of the linear momentum. Whether the magnet is held stationary or not, electromagnetic linear momentum is produced which is equal and opposite to the mechanical linear momentum.

If you let the magnet gain linear momentum, you will need to move to its new rest frame to see its mechanical momentum is zero and the electromagnetic momentum is not. This is what Shockley and James [5] identified as a paradox, but it was really just viewing the system in a rest frame different from that in which the electric field was applied to the magnet.

In this case to start with components containing no momentum you either have to start a current loop moving in the x direction in the lab frame and then apply an electric field in that frame. Or, you can create a current loop in the lab electric field then put it into motion in the x direction. In the former case you give the loop mechanical momentum, then when you apply the electric field in the z direction, you will give the loop an impulse in the positive y direction unless the loop is restrained. The "hidden" momentum will be in the external agent producing the restraint.

It's more complicated when you create the Amperian dipole in an existing electric field. One way to handle this is to create a uniform electric field inside a spherical shell with a dipolar charge distribution and then create a magnetic dipole at the center of the shell [6]. This will give the shell an impulse of $-\boldsymbol{E} \times \boldsymbol{\mu}/c^2$ with an equal and opposite amount of linear momentum in the electromagnetic field inside and outside the shell. No impulse is given to the magnet. The total momentum is zero and no hidden momentum is present.

The question remains, however, since you can model an Amperian dipole in several different ways, which models traveling through the ambient electric field as in Vaidman's paper [13] experience torque? To address this question consider a spherical shell with a surface current density [11] given by

$$\boldsymbol{K} = \sigma \boldsymbol{\omega} \times \boldsymbol{R}. \tag{60}$$

Here, σ is the uniform surface charge density of the charge carriers, $\boldsymbol{\omega} = \omega \hat{\boldsymbol{\imath}}$ is the angular velocity of the charge carriers directed in the positive x direction to conform with the model of the paradox, and $\boldsymbol{R}$ is the position vector from the center of the shell to its surface.

There will be a uniform magnetic field inside the shell given by

$$\boldsymbol{B}_o = \frac{2}{3}\mu_o \sigma R \omega \hat{\boldsymbol{\imath}}, \tag{61}$$

and a dipolar magnetic field outside the shell given by Eq. (25) without the Dirac delta function and where the magnetic moment is

$$\boldsymbol{\mu} = \frac{4\pi}{3}\sigma R^4 \omega \hat{\boldsymbol{\imath}}. \tag{62}$$

An ambient uniform electric field is applied to this magnet, and the Lorentz force that results is offset by an external agent applying an equal and opposite mechanical force. The source of the electric field will also experience a Lorentz force, but it can be considered so massive that it hardly moves. Either that, or the external agent can hold it in place also. There will be linear field momentum equal and opposite to the mechanical momentum.

The usual strategy in special relativistic calculations is to find the reference frame in which the situation is simplest and then Lorentz-transform to the reference frame of interest. Look at the shell in its rest frame and then have the shell move in the positive x direction. Travel along with the shell such that, as the observer, you are also moving through the electric field. This will be the reference frame for computation.

In the slow-motion approximation the ambient electric field is unchanged (multiplied by γ in the fully relativistic case). The fields associated with the shell will be unchanged, since you are moving along with it. This is the case even if the shell has an induced charge due to the ambient electric field. The induced electric field will, in the case of perfect symmetry, be dipolar outside the shell and uniform inside the shell. If the shell is made of conducting material, the electric field of the induced charge inside the shell will be zero.

There will be a magnetic field in your frame of reference due to your motion through the ambient electric field. This field will be

$$\boldsymbol{B}_E = \frac{v}{c^2}E\hat{\boldsymbol{\jmath}}, \tag{63}$$

where E is the magnitude of the ambient field in the z direction. If this field can occupy the same space as the current, there will be a Lorentz force on the current With the geometric relations given, the usual spherical coordinate angles are not convenient. Define two new angles: β will play the part of the polar angle and will be measured from the positive x axis; α will be the azimuth angle and will be measured in the positive direction around the x axis, originating on the positive y axis. (See Figure 6.)

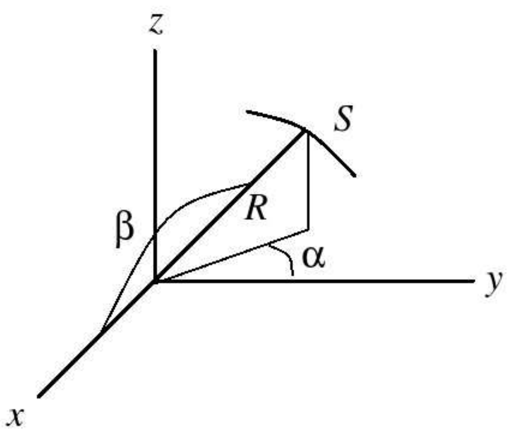

FIG. 6. Angles Used for Calculation. S is the Surface of the Shell.

With this coordinate definition the surface current density will be

$$\boldsymbol{K} = \frac{3\mu}{4\pi R^3}sin\beta\hat{\boldsymbol{\alpha}}, \tag{64}$$

where $\hat{\boldsymbol{\alpha}} = -sin\alpha\hat{\boldsymbol{\jmath}} + cos\alpha\hat{\boldsymbol{k}}$. An element of current dI will be given by the scalar product between the current density and the "area" vector, which in this case will be $d\boldsymbol{l}_\perp = Rd\beta\hat{\boldsymbol{\alpha}}$. So you have

$$dI = \frac{3\mu}{4\pi R^2}sin\beta d\beta. \tag{65}$$

Assuming all the current is in the magnetic field, the force on a current element is given by the usual formula

$$d\boldsymbol{F} = dId\boldsymbol{l} \times \boldsymbol{B}_E = -\frac{3\mu Ev}{4\pi c^2 R}sin^2\beta cos\alpha(d\beta d\alpha)\hat{\boldsymbol{\imath}}. \tag{66}$$

In this case $d\boldsymbol{l} = Rsin\beta d\alpha\hat{\boldsymbol{\alpha}}$ is an element of length along the direction of the current.

It is easy to see that the force integrates to zero due to the $cos\alpha$ dependence. The torque is a different story. It is calculated from the following integral over the surface of the shell.

$$\boldsymbol{\tau} = \int_S \boldsymbol{R} \times (dId\boldsymbol{l} \times \boldsymbol{B}_E = \frac{3\mu Ev}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi} sin^3\beta cos^2(d\beta d\alpha)\hat{\boldsymbol{k}}. \tag{67}$$

Note that the contributions to the torque in the x and y directions integrate to zero. Evaluating the integral gives

$$\boldsymbol{\tau} = \mu \frac{v}{c^2} E v \hat{\boldsymbol{k}}. \tag{68}$$

This is the same result as that found in Eq. (54). However, if the ambient electric field is shielded from the current by conducting material, then $\boldsymbol{B}_E = 0$ there and there will be no torque, something previously pointed out by Franklin [12]. If there is induced charge on the shell that is not shielded from the ambient electric field, the Lorentz force on the two charged hemispheres will be parallel to the z direction and equal and opposite, canceling out and producing no torque but creating stress in the magnet.

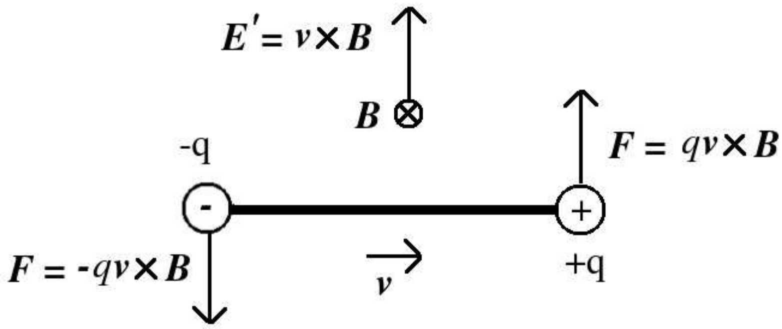

FIG. 7. The Resolution of the "Paradox"

For a magnetic dipole like that of Shockley and James [5], there is no conducting material and the full torque of Eq. (68) should be realized. Since there is no magnetic field in the lab frame in which the shell is moving, how is it that no torque is observed? Actually, the torque is observed as I will now explain.

To get the correct resolution of this paradox, consider the following scenario. Imagine you are an observer in a uniform magnetic field. (See Figure 7.) The magnetic field is into the plane of the figure and there is a nonconducting rod with two equal and opposite charges at the ends moving from left to right, the positive charge in the lead.

An observer moving with the rod sees an electric field due to the Lorentz transformation of the magnetic field. She will see the electric field directed upwards in the figure such that there is an upward force on the positive charge and a downward force on the negative charge. Hence, there will be a torque acting on the rod. You, however, do not detect any electric field. Why then should you see a torque acting on the rod? But you do!

No one to my knowledge considers the above scenario to be a paradox – just the manifestation of the Lorentz force. But the Lorentz force is due to a Lorentz transformation exactly like the case where the current loop is moving through a uniform electric field and experiencing a torque. In other words, this so-called paradox is not a paradox at all. If you Lorentz-transform the torque acting on the magnet in the S' frame to the S (lab) frame, you get the same torque. (In the fully relativistic case the torque is multiplied by γ. However, the mass and therefore the moment of inertia also increases by a factor of γ so the angular motion is the same in both frames.)

This argument brings up another point. It should be clear that you cannot create an interaction in a system where there is none by merely performing a Lorentz transformation. Neither can you Lorentz-transform away an interaction in a system. The interaction involving the Lorentz-transformed field and the magnet in the S' frame cannot be transformed away by observing the system in the S frame.

VIII. AHARONOV-CASHER EFFECT

Tha Aharonov-Bohm effect [24, 25] was a surprising manifestation of the influence of the vector potential of electrodynamics on the quantum behavior of particles not subject to either a magnetic or electric field. It was predicted that two identical charged particles, passing either side of a solenoid would exhibit a phase difference in their wave functions, leading to a detectable interference pattern when the particles interacted after passing the solenoid. This was called a "topological quantum effect".

This was puzzling since there is no electric or magnetic field acting on the particles, and the vector potential field through which the particles traveled was thought by many to be only a mathematical convenience for working out electromagnetic problems. The electric and magnetic fields can be computed from the scalar and vector potentials of electromagnetism, but neither is unique. They can be transformed by what are called gauge transformations and yet yield the same electric and magnetic fields. So it was a surprise that a field that was not considered exactly real could have real effects.

Working with an analogy to the Aharonov-Bohm effect, Aharanov and Casher proposed the same effect would be seen for neutral magnetic particles traveling on either side of, for example, a line of charge [26]. The proposed Aharonov-Casher (AC) effect included the proposition that neutrons would not experience a force while moving in an electric field. Neutrons passing either side of the line of charge with their magnetic moments parallel to the line and to each other would experience unequal phase shifts in their wave functions resulting in a phase difference of

$$\Delta\phi = \mu_o \lambda \mu / \hbar, \tag{69}$$

where λ is the linear charge density, μ is the magnetic moment of the neutron, and $\hbar$ is the reduced Planck's constant. This could appear as a diffraction pattern in an experiment.

Boyer [27] disputed the notion that a neutron in an electric field would not experience a force. Instead, he argued that a moving neutron, modeled as an Amperian magnet, would sport an electric dipole $\boldsymbol{p}$ which would experience a force in an electric field $\boldsymbol{E}$ given by

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}. \tag{70}$$

With an electric field produced by a line of charge,

$$\boldsymbol{E} = \frac{\lambda}{2\pi\epsilon_o r^2}\boldsymbol{r}, \tag{71}$$

where $\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ measured from the line of charge, Boyer computed a force on the neutron given in SI units by

$$\boldsymbol{F} = \frac{\mu_o \mu \lambda v_o}{2\pi r^4}\left[(y^2 - x^2)\hat{\boldsymbol{\imath}} - 2xy\hat{\boldsymbol{\jmath}}\right]. \tag{72}$$

The force is that on a neutron with its magnetic moment parallel to the line of charge (in the positive z direction) moving in the positive y direction with speed v_o. The electric dipole in this case is given by $\boldsymbol{p} = \mu v_o / c^2 \hat{\boldsymbol{\imath}}$.

Boyer considers two such neutrons traveling in the positive y direction with speed v_o, one at $x = +a$ and one at $x = -a$. He assumes the paths will not vary much from straight lines, an assumption that seems justified considering that $\mu_o \mu / 2\pi m = 1.15 \times 10^{-6}$ J·m/A·kg, where m is the neutron mass. He finds that a neutron passing on the positive x side of the wire is delayed with respect to one passing on the negative side by an amount $\Delta y = \mu_o \mu \lambda / m v_o$, which results in the same phase shift as found by Aharonov and Casher in Eq. (69). Hence Boyer claims the phase shift of the AC effect is due to classical lag rather than a quantum topological effect. Arahonov *et al* [28] responded that Boyer overlooked the effect of hidden momentum in the charge-magnet system, which acts to render the net force on the neutron zero.

When Aharonov *et al.* equate the net force acting on the neutron (their equation (6)) to that acting between the line of charge and the induced electric dipole plus that due to the supposed rate of change of the hidden momentum, they find that the net force on the neutron is zero. However, with neither hidden momentum [6, 12, 29, 30] in an charge-magnet system nor an electric dipole on a moving magnetic moment [19, 20], the AC effect needs a different analysis.

You cannot understand the momentum of a charge-magnetic dipole system correctly unless you realize that such a system has to be assembled [6]. A number of researchers have disputed this in personal communications, but certainly there is no reason you cannot assemble such a system. When you apply an electric field to an Amperian magnetic dipole or form such a dipole in a preexisting electric field, mechanical momentum due to Lorentz forces is imparted to both the charge distribution responsible for the electric field and the magnetic dipole. These momenta are equal in magnitude and direction. An opposite amount of momentum is stored in the electromagnetic field. To prevent the components of the system from responding to the Lorentz force, you would need to employ an external agent exerting mechanical forces.

When a point charge is brought into the vicinity of a small Amperian magnet, the magnet gains an amount of mechanical linear momentum $\boldsymbol{\mu} \times \boldsymbol{E}/2c^2$ with an equal momentum gained by the charge, and the opposite of the sum of these is deposited in the electromagnetic field [6]. (The magnet is small in size compared to the variation of the electric field such that the electric field can be considered uniform over the magnet.) The force is due to the magnetic field produced by the displacement current as the charge approaches.

The displacement current will be twice as large for a line of charge approaching a magnet such that the momentum transferred to the magnet is

$$P_m = \boldsymbol{\mu} \times \boldsymbol{E}/c^2, \tag{73}$$

As the magnet moves through the field, the mechanical momentum will in general change with time, giving rise to a force on the magnet. With the electric field given by Eq. (71), the force is given by dP_m/dt with x and y components, respectively,

$$F_x = \frac{\mu_o \mu \lambda}{2\pi r^2}\left[\frac{2}{r^2}(\boldsymbol{r}\cdot\boldsymbol{v})y - \dot{y}\right] = m\ddot{x}, \tag{74}$$

and

$$F_y = \frac{\mu_o \mu \lambda}{2\pi r^2}\left[-\frac{2}{r^2}(\boldsymbol{r}\cdot\boldsymbol{v})x + \dot{x}\right] = m\ddot{y}, \tag{75}$$

the same as found by Boyer. Here $\boldsymbol{v}$ is the velocity of the magnet and m is the magnet's mass. When you make the same assumptions as Boyer (initial velocity $= \dot{y}\hat{\boldsymbol{j}} = v_o\hat{\boldsymbol{j}}$ and $x = \pm a$), these equations become

$$F_x = \frac{\mu_o \mu \lambda}{2\pi r^4}(y^2 - a^2) = m\ddot{x}, \tag{76}$$

and

$$F_y = \frac{\mu_o \mu \lambda}{2\pi r^4}(\pm a y) = m\ddot{y}. \tag{77}$$

From this point on, if you continue the argument of Boyer, you arrive at his result, that the AC phase shift, Eq. (69), is due to a classical lag.

Eqs. (74) and (75) are highly symmetrical and would appear to have solutions $y = y(t)$ and $x = x(t)$ that would be very similar. There are at least two solutions, and one is important for the Aharonov-Casher effect. The trivial solution just has the magnet stationary in the electric field. This would be done my applying mechanical forces to place the magnet at rest in the field.

A more interesting solution can be found by putting Eqs. (74) and (75) in polar coordinates. The equations are then

$$\boldsymbol{F} = -\frac{\mu_o \mu \lambda}{2\pi r^2}(r\dot{\phi}\hat{\boldsymbol{r}} + \dot{r}\hat{\boldsymbol{\phi}}) = m[(\ddot{r} - r\dot{\phi}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\boldsymbol{\phi}}]. \tag{78}$$

Look for a solution where r is constant. The equation reduces to

$$\frac{\mu_o \mu \lambda}{2\pi r}\hat{\boldsymbol{r}} = mr\dot{\phi}\hat{\boldsymbol{r}}. \tag{79}$$

The magnet can therefore execute a circle in the counterclockwise direction (assuming λ is positive) with an angular speed of

$$\omega = \frac{\mu_o \mu \lambda}{2\pi m r^2}. \tag{80}$$

Note that no such orbit exists in the clockwise direction for positive λ.

It is interesting that the circumference of the orbit is exactly the same as the classical lag found by Boyer: $\mu_o\mu\lambda/mv$ where $v = r\omega$. Since this highly improbable orbit (due to the extreme numbers it needs) is due to electromagnetic-derived forces and not to quantum effects, its theoretical existence implies the Aharonov-Casher effect is purely due to a classical lag as Boyer claimed.

IX. THE FORCE BETWEEN TWO MAGNETIC DIPOLES

As many children have learned while playing with bar magnets, there are forces between the poles, either attractive or repulsive. Either way, the forces can do work if the magnets are allowed to move. Mansuripur [31] has called this an apparent violation of the Lorentz force law where a charge is observed to move at a right angle with respect to the Lorentz force as it traverses a magnetic field. No work is done on a body by a force always acting perpendicular to its motion. Since an Amperian magnet consists of moving charges, it seems a paradox that magnets can do work (as they do in many technological applications).

This can be put in relativistic language as follows. Imagine there is a point charge at the origin of its coordinate system S'. The charge is moving with a speed v in the positive x direction in the lab coordinate system S which contains a uniform magnetic field in the positive z direction, $\boldsymbol{B} = B\hat{\boldsymbol{k}}$. In the S' frame there is no current density, but there is a charge. Since the charge is located at $\boldsymbol{r}' = 0$ in S', the current density four-vector is given by

$$J^{\mu'} = (0, 0, 0, cq\delta(\boldsymbol{r}')). \tag{81}$$

In the frame of reference of the charged particle, the magnetic field has been transformed to a slightly modified magnetic field in the same direction $\boldsymbol{B}' = \gamma\boldsymbol{B} \approx \boldsymbol{B}$ and an electric field in the negative y direction, $\boldsymbol{E} = -\gamma(v/c)B\hat{\boldsymbol{j}} \approx -(v/c)B\hat{\boldsymbol{j}}$, where the slow-motion approximation ($\gamma = 1$) has been employed. Have the two frames, S and S' coincide ($\boldsymbol{r} = \boldsymbol{r}'$) at $t = t' = 0$. Application of the Lorentz force law gives the force density four-vector (force acting per unit volume),

$$f^{\mu'} = E^{\mu'\nu'}J_{\nu'} = \begin{pmatrix} 0 & -B & 0 & 0 \\ B & 0 & 0 & -\frac{v}{c}B \\ 0 & 0 & 0 & 0 \\ 0 & \frac{v}{c}B & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ cq\delta(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} 0 \\ -qvB\delta(\boldsymbol{r}) \\ 0 \\ 0 \end{pmatrix}. \tag{82}$$

Integrated over the volume, this is just the same as the more common equation, $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$. The time component of the force four-vector is the time rate change of the internal energy of the charge divided by c and is zero here. The direction of the force is in the y direction, perpendicular to the motion of the charged particle so no work is being done on the charge.

The situation is different if the magnetic field is due to a magnetic dipole and is acting on an Amperian magnet given by a current loop. Consider a current loop in the S' frame with its plane parallel to the y'-z' plane with radius R' and located on the x' axis at $x' > 0$ (Figure 7). The current in the loop is taken to be counterclockwise about the positive x' direction such that its current density at R' is

$$J^{\mu'} = (0, J_{y'}, J_{z'}, 0) = (0, -J' sin\theta', J' cos\theta', 0). \tag{83}$$

Have a magnetic dipole on the x' axis with its magnetic moment $\boldsymbol{\mu}_D$ pointing in the positive x' direction and moving toward the current loop at speed v. Also, employ the slow-motion approximation so that $x' = x$, $t' = t$, $J' = J$, and $\theta' = \theta$ (and, of course, $y' = y$, $z' = z$, and $R' = R$), then in the S' frame the magnetic field is the same as in the S frame, given by

$$\begin{aligned} B_x &= \frac{\mu_o\mu_D}{4\pi r^5}\left[3z^2 - r^2\right], \\ B_y &= \frac{3\mu_o\mu_D}{4\pi r^5}xy, \\ B_z &= \frac{3\mu_o\mu_D}{4\pi r^5}xz. \end{aligned} \tag{84}$$

Plus there is an electric field given by

$$\begin{aligned} E_x &= 0, \\ E_y &= -\frac{3\mu_o\mu_D v}{4\pi r^5}xz, \\ E_z &= \frac{3\mu_o\mu_D v}{4\pi r^5}xy, \end{aligned} \tag{85}$$

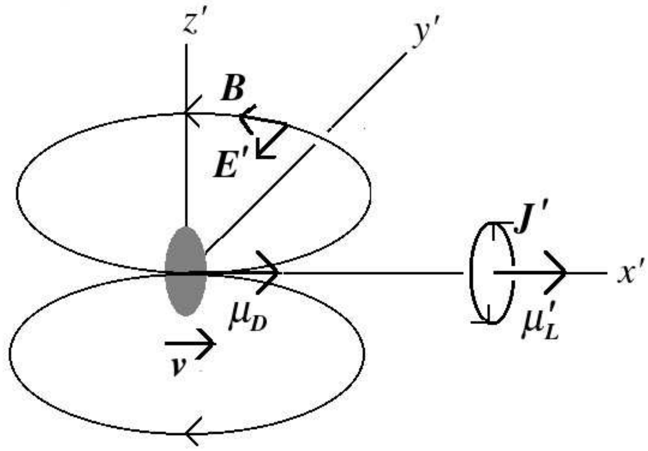

FIG. 8. Two Interacting Magnets, North and South Poles Facing Each Other

in the S' frame due to the Lorentz transformation between frames. (In Figure 8 two magnetic field line loops of the dipole are shown with a representative $\boldsymbol{B}$ and $\boldsymbol{E}$ field at a point on one loop.) The following equation finds the force density four-vector.

$$f^{\mu'} = E^{\mu'\nu'} J_{\nu'} = \frac{\mu_o \mu_D}{4\pi r^5} \begin{pmatrix} 0 & -3xz & 3xy & 0 \\ 3xz & 0 & -3x^2 + r^2 & -3\frac{v}{c}xz \\ -3xy & 3x^2 - r^2 & 0 & 3\frac{v}{c}xy \\ 0 & -3\frac{v}{c}xz & 3\frac{v}{c}xy & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -J_y \\ -J_z \\ 0 \end{pmatrix}$$

$$= \frac{\mu_o \mu_D}{4\pi r^5} \begin{pmatrix} 3x(zJ_y - yJ_z) \\ (3x^2 - r^2)J_z \\ -(3x^2 - r^2)J_y \\ 3(v/c)x(zJ_y - yJ_z) \end{pmatrix}. \tag{86}$$

The components of the force density four-vector are

$$\begin{aligned} f_x &= -\frac{3\mu_o \mu_D}{4\pi r^5} xR\rho u \\ f_y &= \frac{3\mu_o \mu_D}{4\pi r^5}(3x^2 - r^2)\rho u cos\theta = \frac{3\mu_o \mu_D}{4\pi r^5}(2x^2 - R^2)\rho u cos\theta \\ f_z &= -\frac{3\mu_o \mu_D}{4\pi r^5}(3x^2 - r^2)\rho u sin\theta = -\frac{3\mu_o \mu_D}{4\pi r^5}(2x^2 - R^2)\rho u sin\theta \\ f_{ct} &= \frac{v}{c} f_x, \end{aligned} \tag{87}$$

where the current density equals the charge density ρ times the current drift speed u. When you integrate these force densities over the volume of the loop, the forces parallel to the y and z directions are zero due to the sinusoidal functions. (This can also be inferred by symmetry.) The torque on the loop is also zero in contrast to the conclusion by Mansuripur [31] due to the fact the forces in the y and z direction are either toward or away from the center of the dipole. (His result was due to taking the loop to be charged, but, of course, most magnets are neutral).

The case for the force parallel to the x direction is different. The volume integral is

$$\begin{aligned} F_x &= \int f_x dV \\ &= -\frac{3\mu_o\mu_D}{4\pi r^5} xuR \int \rho dV \\ &= -\frac{3\mu_o\mu_D}{4\pi r^5} xuR\lambda \int_0^{2\pi R} dl \\ &= -\frac{3\mu_o\mu_D\mu_L}{2\pi r^4}. \end{aligned} \tag{88}$$

The final result is gotten from the following: ρdV is an element of the charge of the current in the loop and equals λdl, where λ is the charge per unit length, and $dl = Rd\theta$ is an element of the loop's circumference. Also, the current $I = \lambda u$ and the magnetic moment of the loop is $\mu_L = I\pi R^2$. Finally you let $R \to 0$ and $x \to r$ as μ_L is held constant, forming a point dipole.

You see from the fact that there is a force given by Eq. (88) acting on the current loop in the same direction as its motion (or in the opposite direction if like poles face each other) that work can be done by the magnetic field. (Even if you never played with magnets!) Mansuripur argued that torque acting on the electric charge of the current does work equal and opposite to the work done by the force in Eq. (88) such that there is no net work done. However, a magnet typically is uncharged; there is as much negative charge as there is positive charge. In this case there is no torque and Manuripur's argument fails.

Of course, each moving charge in the current loop experiences a Lorentz force at right angles to its motion. The answer to this apparent paradox lies in the fact that the charges cannot move freely in the magnetic field. This is mathematically embodied in the expression for the current density in Eq. (83). If the charges could move freely, the current in the loop would be dispersed as the charges began to follow looping paths in the magnetic field. Instead, they are confined to move as implied in Eq. (83) and so produce a force on the loop as they are scattered by phonons, impurities, and lattice defects.

The time rate of change of the internal energy of the magnet due to the force F_x is the time component of the force four-vector divided by c, that is,

$$\frac{d\mathcal{E}}{dt} = \frac{v}{c^2} F_x = -\frac{v}{c}\frac{3\mu_o\mu_D\mu_L}{2\pi r^4}, \tag{89}$$

where $\mathcal{E}$ is the internal energy. Since the Lorentz force preserves rest mass, the negative of this is the time rate of change of the work done by the force, which can be seen by calculating the power from force times velocity, vF_x. The energy for doing this work must come from the magnetic field as there is no other source of energy available. The time-dependent energy density of the magnetic field is $\boldsymbol{B}_D \cdot \boldsymbol{B}_L/\mu_o$, where $\boldsymbol{B}_D$ and $\boldsymbol{B}_L$ are the magnetic fields of the magnetic dipole and current loop, respectively. As the magnets are pulled together, which is the case for the above calculations, the fields between the poles tend to cancel, lowering the field energy. When like poles repel, once again energy is tapped in like manner from the magnetic field.

In both the S and S' frames there will be an electric field due to the Lorentz transformation given by Eq. (85). Thus there will be a field momentum density given by $\epsilon_o \boldsymbol{E} \times \boldsymbol{B}$. It is easy to see, however, that this density integrates to zero over the volume of the dipole-dipole system, since all of the integrands are odd in the coordinates. Hence, there is no net electromagnetic field momentum.

X. SUMMARY

The understanding of momentum in charge-magnet systems has been hampered by not taking the formation of these systems into account. When an Amperian magnet is subjected to an electric field or is formed in a preexisting electric field (or some combination thereof), Lorentz forces arise that impart momentum to the magnet and the charges or to an external agent exerting mechanical forces on them. An equal and opposite amount of momentum is added to the electromagnetic field. The solution to the Shockley-James paradox [5] is that their charge-magnet system is either not being viewed in its original rest frame or the mechanical momentum that the system would gain is present in an external agent. No hidden momentum resides in the charge-magnet system [6].

It has generally been thought that an Amperian magnet moving in an observer's frame of reference will be observed to have an electric dipole present on it perpendicular to both the magnetic moment and the direction of motion. However, there is no such dipole; its mathematical manifestation is due to ignoring the effects of the relativity of simultaneity [19, 20].

The solution to the paradox of Mansuripur [16] is simply that there is no electric dipole on a moving magnetic dipole. Hence the supposed torque that would be seen on a moving charge-magnet system due to the interaction of the charge with the electric dipole is not present. No torque is seen whether or not the system is in motion [32].

As Furry [11] has shown, a charge-magnet system where the magnet is Amperian can contain both linear and angular momentum in its electromagnetic field. This momentum is obtained when the system is formed and balances the mechanical momentum that is also generated [6]. When the system is in motion there is a torque, but it is in the electromagnetic field and has been mistaken as a mechanical torque acting on the magnet.

There are two equal and opposite sources of torque in a moving charge-magnet system, yielding no net torque. One is due to the motion of the electromagnetic field. The field angular momentum Furry identified is Lorentz-transformed into a time-dependent angular momentum, thus producing a torque. The other source is due to the interaction between the charge and the current loop of the magnet, not between the charge and an electric dipole on the magnet [32].

A current loop can experience a torque in a magnetic field unless the field and magnetic moment are parallel or antiparallel. A current loop moving through an electric field will in general be subject to a Lorentz-transformed magnetic field and thus should experience a torque. An observer at rest with the electric field sees no magnetic field, and it has been thought that the observer would detect no torque, thus creating a paradox. Vaidman [13] appeared to have solved this paradox by calculating the torque to be zero with hidden momentum in the current loop and by assuming the torque in the frame of the at rest observer to be zero also.

However, if there actually is torque acting on the current loop in its frame of reference, a Lorentz transformation to any other frame of reference cannot do away with that torque. So, if there is a torque observed in the moving frame due to a magnetic field in that frame, there will also be a torque seen in any other reference frame, whether or not a magnetic field is detected in that frame. Thus no paradox exists here.

In the Aharonov-Casher effect [26] neutral Amperian magnets (such as exist on neutrons) are supposed to experience a differential phase shift in their wave functions as they pass on either side of a line of charge. According to this effect, the phase shift difference is a result of what is called a quantum topological effect rather than some force that causes one magnet to beat another to a detection system. Boyer [27] disputed the quantum nature of the effect by calculating a lag between neutrons passing either side of the line of charge due to a force between the charge of the line and an electric dipole on the moving magnet. However there is no electric dipole present and this explanation does not work.

Aharonov *et al.* [28] accepted the existence of the force Boyer identified but argued that hidden momentum was involved in canceling it. There is no hidden momentum and no electric dipole, but there is still a force on the moving magnet due to the change in its mechanical momentum to cancel the opposite change in the electromagnetic momentum. As such, there is a lag between magnets moving on opposite sides of the line of charge, and this lag turns out to be the same as that calculated by Boyer.

Mansuripur [31] addressed the force between two magnets as a paradox due to the fact that the force can do work. However, the Lorentz force on a charge moving through a magnetic field is always perpendicular to the direction of motion and thus cannot do work. This is not really a paradox. The charges in an Amperian magnet subjected to a magnetic field are not able to move freely but are kept on a track determined by the conducting material. Or, if the magnet is made up of counter-rotating non-conducting disks such as that of Shockley and James [5], the charges are fixed in the disks and are not free to move in response to the Lorentz force. The force on the magnet in either case is due to the interaction of the charges with the confining material.

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[1] J. J. Thomson, Phil. Mag. **31**, 149 (1891).
[2] J. D. Jackson, *Classical Electrodynamics* (John Wiley & sons, New York, 1962).
[3] W. Rindler, *Relativity: Special, General, and Cosmological*, 2nd ed. (Oxford University Press, Oxford, 2006).
[4] A. Einstein, Annalen der Physik **17**, 891 (1905).
[5] W. Shockley and R. P. James, Phys. Rev. Lett **18**, 876 (1967).
[6] F. R. Redfern, Eur. Phys. J. D. **71**, 163 (2017).
[7] D. Babson *et al.*, Am. J. Phys. **77**, 826 (2009).
[8] M. Planck, Phys. ZS **9**, 828 (1909).

[9] M. von Laue, Verhandlungen der Deutschen Physikalischen Gesellschaft **13**, 513 (1911), english translation: https://en.wikisource.org/?curid=1151856.
[10] M. Janssen, *A comparison between lorentzs ether theory and Special Relativity in the light of the experiments of Trouton and Noble*, Ph.D. thesis, University of Pittsburgh (1995).
[11] W. H. Furry, Am. J. Phys. **37**, 621 (1969).
[12] J. Franklin, Eur. J. Phys. **39**, 035201 (2016).
[13] L. Vaidman, Am. J. Phys. **58**, 978 (1990).
[14] V. Hnizdo, Am. J. Phys. **60**, 279 (1992).
[15] A. Cho, Science **336**, 404 (2012).
[16] M. Mansuripur, Phys. Rev. Lett. **108**, 193901 (2012).
[17] A. Einstein and J. Laub, Ann. Phys. **331**, 541 (1908).
[18] D. J. Griffiths and V. Hnizdo, Am. J. Phys. **81**, 570 (2013).
[19] J. Franklin, http://arXiv:1603.02912 [physics.gen-ph] (2018).
[20] F. R. Redfern, Can. J. Phys. **97**, 125 (2019).
[21] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Dover Publications, Mineola, New York, 2005) p. 1.
[22] D. Bedford and P. Krumm, Am. J. Phys. **54**, 1036 (1986).
[23] V. Namias, Am. J. Phys. **57**, 171 (1989).
[24] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[25] R. G. Chambers, Phys. Rev. Lett. **5**, 3 (1960).
[26] Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).
[27] T. H. Boyer, Phys. Rev. A **36**, 5083 (1987).
[28] Y. Aharonov, P. Pearle, and L. Vaidman, Phys. Rev. A **37**, 4052 (1988).
[29] J. Franklin, Am. J. Phys. **82**, 869 (2014).
[30] F. R. Redfern, Eur. Phys. J. D **71**, 325 (2017).
[31] M. Mansuripur, Spintronics X, Proceedings of SPIE **10357**, 10357 (2017).
[32] F. R. Redfern, Phys. Scr. **91**, 045501 (2016).